QUANTUM COHOMOLOGY OF G/P AND HOMOLOGY OF AFFINE GRASSMANNIAN

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Abstract. Let G be a simple and simply-connected complex algebraic group, $P \subset G$ a parabolic subgroup. We prove an unpublished result of D. Peterson which states that the quantum cohomology $QH^*(G/P)$ of a flag variety is, up to localization, a quotient of the homology $H_*(\mathrm{Gr}_G)$ of the affine Grassmannian Gr_G of G. As a consequence, all three-point genus zero Gromov-Witten invariants of G/P are identified with homology Schubert structure constants of $H_*(\mathrm{Gr}_G)$, establishing the equivalence of the quantum and homology affine Schubert calculi.

For the case $G = B$, we use the Mihalcea's equivariant quantum Chevalley formula for $QH^*(G/B)$, together with relationships between the quantum Bruhat graph of Brenti, Fomin and Postnikov and the Bruhat order on the affine Weyl group. As byproducts we obtain formulae for affine Schubert homology classes in terms of quantum Schubert polynomials. We give some applications in quantum cohomology.

Our main results extend to the torus-equivariant setting.

1. Introduction

Let G be a simple and simply-connected complex algebraic group, $P \subset G$ a parabolic subgroup and T a maximal torus. This paper studies the relationship between the quantum cohomology $QH^*(G/P)$ of the flag variety of G and the homology $H_*(\mathrm{Gr}_G)$ of the affine Grassmannian Gr_G of G. We show that $QH^*(G/P)$ is a quotient of $H_*(\mathrm{Gr}_G)$ after localization and describe the map explicitly on the level of Schubert classes. As a consequence, all three-point genus zero Gromov-Witten invariants of G/P are identified with homology Schubert structure constants of $H_*(\mathrm{Gr}_G)$, establishing the equivalence of the quantum and homology affine Schubert calculi. This is an unpublished result stated by Dale Peterson in 1997 [22]. Peterson's statement and our proof extends to the T-equivariant setting, though Peterson was not using the definition of equivariant quantum cohomology in use today.

Quantum Schubert calculus has been studied heavily and we will not attempt to survey the literature. The combinatorial study of the *equivariant* quantum cohomology rings $QH^T(G/P)$ is however more recent (see [21]). Schubert calculus on the affine Grassmannian was first studied by Kostant and Kumar [13] as a special case of their general study of the topology of Kac-Moody flag varieties. That the nilHecke ring of Kostant and Kumar could be used to study both the homology and cohomology of the affine Grassmannian was first realized by Peterson, who should be considered the father of affine Schubert calculus. Peterson's work on affine Schubert calculus is related to his theory of geometric models for $QH^*(G/P)$, most of

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which has remained unpublished for a decade; see however [12] and [24] for statements of some of Peterson's results. Recently, interest in affine Schubert calculus was rekindled from a different direction: Shimozono conjectured and later Lam [14] proved that the k-Schur functions of Lapointe, Lascoux and Morse [17], arising in the study of Macdonald polynomials, represented homology Schubert classes of the affine Grassmannian when $G = SL(n)$.

The observation that $QH^*(G/P)$ and $H_*(\mathrm{Gr}_G)$ are related, is already apparent in the literature. Ginzburg [9] described the cohomology $H^*(\mathrm{Gr}_G)$ as the enveloping algebra of the Lie algebra of a unipotent group. The same unipotent group occurs in Kostant's [12] description of $QH^*(G/B)$ as a ring of rational functions. More recently, Bezrukavnikov, Finkelberg and Mirkovic [2] described the equivariant K-homology of Gr_G and discovered a relation with the Toda lattice. Earlier the relation of the Toda lattice with $QH^*(G/B)$ had been established by Kim [11]. One can already deduce from [2] and [11] that some localizations of $H_*(\mathrm{Gr}_G)$ and $QH^*(G/B)$ are isomorphic[1]. However, such a statement is insufficient for the enumerative applications to Schubert calculus. On the other hand, even knowing the coincidence of Gromov-Witten invariants with affine homology Schubert structure constants, the fact that the identification arises from a ring homomorphism is still unexpected; for example, the theorems of [4, 25] which compare structure constants in quantum and ordinary cohomology, are not of this form. However we note that Lapointe and Morse [19] defined a ring homomorphism from the linear span of k-Schur functions to the quantum cohomology of the Grassmannian, which via [14] may be interpreted as sending Schubert classes in the homology of the affine Grassmannian of SL_{k+1} to quantum Schubert classes.

The paper is naturally separated into the two cases $P = B$ and $P \neq B$. For $P = B$, our proof is purely algebraic and combinatorial, and does not appeal to geometry as in (what we believe is) Peterson's original intended argument, though much of the combinatorics we develop may well have been known to Peterson. At the core of the our argument is the relationship between the *quantum Bruhat graph*, first studied by Brenti, Fomin and Postnikov [3] and the Bruhat order on the *superregular* elements of the affine Weyl group, which we study here. Roughly speaking, an element x of the affine Weyl group W_{af} is superregular if it has a large translation component. As a byproduct, we show that the *tilted Bruhat orders* in [3] are all (dual to) induced suborders of the affine Bruhat order.

The algebraic part of our proof relies on known properties of the ring $QH^*(G/B)$, in particular the fact that it is associative and commutative. Apart from these general properties, we need only one more formula for $QH^*(G/B)$: the equivariant quantum Chevalley formula originally stated by Peterson [22], and recently proved by Mihalcea [21]. On the side of $H_*(\mathrm{Gr}_G)$, our computations rely on a homomorphism $j : H_T(\mathrm{Gr}_G) \to Z_{\mathbb{A}_{\mathrm{af}}}(S) \subset \mathbb{A}_{\mathrm{af}}$, where $\mathbb{A}_{\mathrm{af}}$ is the affine nil Hecke ring of Kostant and Kumar [13] and $Z_{\mathbb{A}_{\mathrm{af}}}(S)$ (called the *Peterson subalgebra* in [14]) is the centralizer of $S = H^T(\mathrm{pt})$. The map j is again due to Peterson. Proofs of its main properties can be found in [14].

Our results allow us to give formulae for the affine Schubert classes as elements of the Peterson subalgebra. These formulae involve generating functions over paths in the affine Bruhat order, or equivalently in the quantum Bruhat graph. In particular, our formulae are related to the quantum Schubert polynomials of [7, 20]. Each

[1]Finkelberg (private communication) has calculated these localizations in the context of [2].

quantum Schubert polynomial gives a formula for infinitely many affine Schubert classes.

For the case $P \neq B$ we study the Coxeter combinatorics of the affinization of the Weyl group of the Levi factor of P. We use this combinatorics to compare the quantum equivariant Chevalley formulae for $QH^T(G/B)$ and $QH^T(G/P)$, using the comparison formula of Woodward [25] to refine the Chevalley formula of [8, 21]. Some of the intermediate results we use are stated by Peterson in [22].

We use the affine homology Chevalley formula given in [16] to deduce a formula in $QH^*(G/P)$ for multiplication by the quantum Schubert class $\sigma_P^{r_\theta}$ labeled by the reflection r_θ in the highest root. We show that in the case of the Grassmannian, the ring homomorphism of Lapointe and Morse [19] differs from Peterson's map by the *strange duality* of $QH^*(G/P)$ due to Chaput, Manivel and Perrin [6].

In the current work we use the maximal torus T in G; yet the affine Grassmannian affords the additional $\mathbb{C}^*$-action given by loop rotation. In future work we intend to study the Schubert calculus of the affine Grassmannian with respect to this extra $\mathbb{C}^*$-equivariance and to pursue K-theoretic analogues of Peterson's theory.

Both the quantum cohomology $QH^*(G/B)$ and homology $H_*(\mathrm{Gr}_G)$ possess additional structures which would be interesting to compare: for example, $QH^*(G/B)$ has mirror-symmetric constructions and $H_*(\mathrm{Gr}_G)$ is a Hopf algebra with an action of the nilHecke ring. The naturality of our main theorem with respect to Schubert classes suggests that the appearance of the Toda Lattice in [2, 11] is somehow related to Schubert calculus.

2. The equivariant quantum cohomology ring $QH^T(G/B)$

2.1. **Notations.** Let G be a simple and simply-connected complex algebraic group, $B \subset G$ a Borel subgroup and $T \subset B$ a maximal torus. Let $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ be a basis of simple roots and $\{\alpha_i^\vee \mid i \in I\} \in \mathfrak{h}$ a basis of simple coroots, where $\mathfrak{h}$ is a Cartan subalgebra of the Lie algebra of G. Denote by $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ and $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$ the root and coroot lattices. Let $P = \bigoplus_{i\in I} \mathbb{Z}\omega_i \subset \mathfrak{h}^*$ and $P^\vee = \bigoplus_{i\in I} \mathbb{Z}\omega_i^\vee \subset \mathfrak{h}$ be the weight and coweight lattices, where $\{\omega_i \mid i \in I\}$ and $\{\omega_i^\vee \mid i \in I\}$ are the fundamental weights and coweights, which are the dual bases to $\{\alpha_i^\vee \mid i \in I\}$ and $\{\alpha_i \mid i \in I\}$ with respect to the natural pairing $\langle \cdot\,,\,\cdot\rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$.

Let W denote the Weyl group; it is generated by the simple reflections $\{r_i \mid i \in I\}$. Let $\ell : W \to \mathbb{Z}$ denote the length function of W. We let $w < v$ denote a relation in the Bruhat order of W and write $w \lessdot v$ if $w < v$ and $\ell(w) = \ell(v) - 1$. W acts on $\mathfrak{h}^*$ and $\mathfrak{h}$ by

$$\begin{aligned} r_i\mu &= \mu - \langle \alpha_i^\vee\,,\,\mu\rangle\alpha_i \qquad &&\text{for } \mu \in \mathfrak{h}^* \\ r_i\lambda &= \lambda - \langle \lambda\,,\,\alpha_i\rangle\alpha_i^\vee \qquad &&\text{for } \lambda \in \mathfrak{h}. \end{aligned}$$

These actions stabilize the lattices $Q \subset P \subset \mathfrak{h}^*$ and $Q^\vee \subset P^\vee \subset \mathfrak{h}$ respectively. The pairing $\langle \cdot\,,\,\cdot\rangle$ is W-invariant: for all $w \in W$, $\mu \in \mathfrak{h}^*$, and $\lambda \in \mathfrak{h}$, we have

$$\langle w \cdot \lambda\,,\, w \cdot \mu\rangle = \langle \lambda\,,\,\mu\rangle.$$

Let $R = W \cdot \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ be the root system of G. Then $R = R^+ \sqcup -R^+$ where $R^+ = R \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ is the set of positive roots. For each $\alpha \in R$ there is a $u \in W$ and $i \in I$ such that $\alpha = u\alpha_i$. Define the associated coroot $\alpha^\vee \in Q^\vee$ of α by $\alpha^\vee = u\alpha_i^\vee$ and the associated reflection of α by $r_\alpha = ur_iu^{-1} \in W$; they are independent of the choice of u and i.

2.2. **Quantum equivariant Chevalley formula.** Let us denote by $S = H^T(\mathrm{pt})$ the symmetric algebra of the weight lattice P. Let $\mathbb{Z}[q] = \mathbb{Z}[q_i \mid i \in I]$ be a polynomial ring for the sequence of indeterminates q_i. For $\lambda = \sum_{i\in I} a_i\, \alpha_i^\vee \in Q^\vee$ with $a_i \in \mathbb{Z}_{\geq 0}$ we set $q_\lambda = \prod_{i\in I} q_i^{a_i} \in \mathbb{Z}[q]$. The (small) equivariant quantum cohomology $QH^T(G/B)$ is isomorphic to $H^T(G/B) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a free $\mathbb{Z}[q]$-module, with basis the equivariant quantum Schubert classes $\{\sigma^w \in QH^T(G/B) \mid w \in W\}$. It is equipped with a quantum multiplication denoted $* : QH^T(G/B) \times QH^T(G/B) \to QH^T(G/B)$. This multiplication is associative and commutative.

When we set $q_i = 1$ in $QH^T(G/B)$ we obtain the usual equivariant cohomology $H^T(G/B)$. When we apply the evaluation $\phi_0 : S \to \mathbb{Z}$ at 0 to $QH^T(G/B)$ we obtain the usual quantum cohomology $QH^*(G/B)$. We refer the reader to [21] for more details. As shown in [21], the *quantum equivariant Chevalley formula* completely determines the multiplication in $QH^T(G/B)$. It was first stated by Peterson [22] and proved by Mihalcea [21]. Define the element $\rho = \sum_{i\in I} \omega_i = \frac{1}{2}\sum_{\alpha\in R^+} \alpha \in P$.

Theorem 2.1 (Quantum equivariant Chevalley formula)**.** *Let $i \in I$ and $w \in W$. Then we have in $QH^T(G/B)$*

$$\sigma^{r_i} * \sigma^w = (\omega_i - w\cdot\omega_i)\sigma^w + \sum_\alpha \langle \alpha^\vee, \omega_i\rangle \sigma^{wr_\alpha} + \sum_\alpha \langle \alpha^\vee, \omega_i\rangle q_{\alpha^\vee} \sigma^{wr_\alpha}$$

where the first summation is over $\alpha \in R^+$ such that $wr_\alpha \gtrdot w$ and the second summation is over $\alpha \in R^+$ such that $\ell(wr_\alpha) = \ell(w) + 1 - \langle \alpha^\vee, 2\rho\rangle$.

Our notation here differs slightly from Mihalcea's: the indexing of Schubert bases has been changed via $w \mapsto w_0 w$, and we have made a different choice of positive roots for T. However, our indexing agrees with the ones in [7, 8, 13].

Theorem 2.1 can be extended by linearity to give a formula for the multiplication by the quantum equivariant class $[\lambda] \in QH^T(G/B)$ of a line bundle with weight λ. Theorem 2.1 then corresponds to the case $\lambda = \omega_i$. Let us denote by $c_{u,v}^{w,\lambda} \in S$ the equivariant Gromov-Witten invariants given by

$$\sigma^v * \sigma^u = \sum_{w\in W} c_{u,v}^{w,\lambda} q_\lambda\, \sigma^w$$

in $QH^T(G/B)$. The non-equivariant Gromov-Witten invariants have an explicit enumerative interpretation which we will not describe here.

2.3. **Quantum Bruhat graph.** The *quantum Bruhat graph* $D(W)$ of [3] is the directed graph with vertices given by the elements of the Weyl group W, with a directed edge from w to $v = wr_\alpha$ for $w \in W$ and $\alpha \in R^+$ if either $\ell(v) = \ell(w) + 1$ or $\ell(v) = \ell(w) + 1 - \langle \alpha^\vee\, , 2\rho\rangle$.

Given $u \in W$, the *tilted Bruhat order* $D_u(W)$ of [3] is the graded partial order on W with the relation $w \prec_u v$ if and only if there is a shortest path in $D(W)$ from u to v which passes through w. Note that $D_{\mathrm{id}}(W)$ is the usual Bruhat order. We refer the reader to [3, Section 6] for further details.

3. Affine weyl group

Let $W_{\mathrm{af}} = W \ltimes Q^\vee$ denote the affine Weyl group corresponding to W. For $\lambda \in Q^\vee$, its image in W_{af} is denoted t_λ. We have $t_{w\cdot\lambda} = wt_\lambda w^{-1}$ for all $w \in W$ and $\lambda \in Q^\vee$. As a Coxeter group W_{af} is generated by simple reflections $\{r_i \mid i \in I_{\mathrm{af}}\}$ where $I_{\mathrm{af}} = I \sqcup \{0\}$. We denote by $Q_{\mathrm{af}} = \oplus_{i\in I_{\mathrm{af}}} \mathbb{Z}\alpha_i \subset \mathfrak{h}_{\mathrm{af}}^*$ and $Q_{\mathrm{af}}^\vee = \oplus_{i\in I_{\mathrm{af}}} \mathbb{Z}\alpha_i^\vee \subset$

$\mathfrak{h}_{\mathrm{af}}$ the affine root and coroot lattices, where $\mathfrak{h}_{\mathrm{af}}$ is the Cartan subalgebra of the affine Lie algebra $\mathfrak{g}_{\mathrm{af}}$ associated to the Lie algebra $\mathfrak{g}$ of G. Restriction yields a natural map $Q_{\mathrm{af}} \to Q$ denoted $\beta \mapsto \bar{\beta}$; its kernel is spanned by the null root $\delta = \sum_{i \in I_{\mathrm{af}}} a_i \alpha_i = \alpha_0 + \theta$ where $\theta \in R$ is the highest root. In particular we have $Q_{\mathrm{af}} \cong Q \oplus \mathbb{Z}\delta$. Abusing notation we sometimes write α both for an element of Q_{af} and its image $\bar{\alpha}$ in Q.

The affine root system R_{af} is comprised of the nonzero elements of the form $\beta = \alpha + n\delta$ where $\alpha \in R \cup \{0\}$ and $n \in \mathbb{Z}$. The set of positive affine roots R_{af}^+ consists of the elements $\alpha + n\delta \in R_{\mathrm{af}}$ such that either $n > 0$ or both $\alpha \in R^+$ and $n = 0$.

Let $R_{\mathrm{af}}^{\mathrm{re}} = W_{\mathrm{af}} \cdot \{\alpha_i \mid i \in I_{\mathrm{af}}\}$ be the set of real roots of $\mathfrak{g}_{\mathrm{af}}$; it consists of the elements $\beta \in R_{\mathrm{af}}$ such that $\bar{\beta} \neq 0$. The associated coroot of $\beta \in R_{\mathrm{re}}^{\mathrm{af}}$ is defined by $\beta^\vee = u\alpha_i^\vee \in Q_{\mathrm{af}}^\vee$ where $u \in W_{\mathrm{af}}$ and $i \in I_{\mathrm{af}}$ are such that $\beta = u\alpha_i$; $\beta^\vee$ is independent of the choice of u and i. The associated reflection is defined by $r_\beta = u r_i u^{-1} \in W_{\mathrm{af}}$.

The level zero action of W_{af} on $P \oplus \mathbb{Z}\delta$ is given by

$$wt_\lambda \cdot (\mu + n\delta) = w \cdot \mu + (n - \langle \lambda \, , \, \mu \rangle)\delta \tag{1}$$

for $w \in W$, $\lambda \in Q^\vee$, $\mu \in P$ and $n \in \mathbb{Z}$. This action stabilizes Q_{af}. For $\beta = \alpha + n\delta \in R_{\mathrm{af}}^{\mathrm{re}}$, with respect to $W_{\mathrm{af}} = W \ltimes Q^\vee$ one has

$$r_\beta = r_\alpha t_{n\alpha^\vee} \tag{2}$$

and, in particular,

$$r_0 = r_\theta t_{-\theta^\vee}.$$

For $x \in W_{\mathrm{af}}$, define

$$\mathrm{Inv}(x) = \{\beta \in R_{\mathrm{af}}^+ \mid x \cdot \beta \in -R_{\mathrm{af}}^+\};$$

the elements of $\mathrm{Inv}(x)$ are called *inversions* of x. It is well-known that $\ell(x) = |\mathrm{Inv}(x)|$ for all $x \in W_{\mathrm{af}}$. The following standard formula gives the length of $x = wt_\lambda$. It is obtained by calculating the number of values of $n \in \mathbb{Z}$, for each fixed $\alpha \in R^+$ such that $\alpha + n\delta \in \mathrm{Inv}(x)$.

Lemma 3.1. *Let $x = wt_\lambda \in W_{\mathrm{af}}$. Then*

$$\ell(x) = \sum_{\alpha \in R^+} |\chi(w \cdot \alpha < 0) + \langle \lambda, \alpha \rangle|$$

where $\chi(P) = 1$ if P is true and $\chi(P) = 0$ otherwise.

We call $\lambda \in \mathfrak{h}$ *antidominant* if $\langle \lambda \, , \, \alpha_i \rangle \leq 0$ for each $i \in I$, and denote by $\tilde{Q}$ the set of antidominant elements of $Q^\vee$. The following lemma is an immediate consequence of Lemma 3.1.

Lemma 3.2. *Let $\lambda \in Q^\vee$ and $w \in W$ such that $w \cdot \lambda \in \tilde{Q}$. Then $\ell(t_\lambda) = \langle w \cdot \lambda \, , \, -2\rho \rangle$.*

Let W_{af}^- denote the set of *Grassmannian* elements in W_{af}, which by definition are those that are of minimum length in their coset in W_{af}/W. They are characterized below.

Lemma 3.3. *Let $w \in W$ and $\lambda \in Q^\vee$. Then $wt_\lambda \in W_{\mathrm{af}}^-$ if and only if $\lambda \in \tilde{Q}$ and w is λ-minimal, that is, for every $i \in I$, if $\langle \lambda \, , \, \alpha_i \rangle = 0$ then $w\alpha_i > 0$ (equivalently, w is of minimum length in its coset in W/W_λ where W_λ is the stabilizer of λ). In this case $\ell(wt_\lambda) = \ell(t_\lambda) - \ell(w)$.*

Proof. We have $wt_\lambda \in W_{\mathrm{af}}^-$ if and only if $wt_\lambda \cdot \alpha_i > 0$ for each $i \in I$. By (1) this holds if and only if for each $i \in I$ either $\langle \lambda \,,\, \alpha_i \rangle < 0$ or $\langle \lambda \,,\, \alpha_i \rangle = 0$ and $w \cdot \alpha_i \in R^+$. This is exactly the stated condition. To calculate $\ell(wt_\lambda)$ in this case one observes that for each $\alpha \in R^+$ we have $\chi(w \cdot \alpha < 0) + \langle \lambda \,,\, \alpha \rangle \leq 0$, so by Lemma 3.1, $\ell(t_\lambda) - \ell(wt_\lambda)$ is equal to the number of inversions of w. $\square$

We say that $\lambda \in Q^\vee$ is regular if the stabilizer W_λ is trivial.

Lemma 3.4. *For $\lambda \in \tilde{Q}$ regular,*

$$\ell(ut_{w\cdot\lambda}) = \ell(t_\lambda) - \ell(uw) + \ell(w).$$

Proof. We have $ut_{w\cdot\lambda} = uwt_\lambda w^{-1}$. By Lemma 3.3, $uwt_\lambda \in W_{\mathrm{af}}^-$ and $\ell(uwt_\lambda) = \ell(t_\lambda) - \ell(uw)$. But $\ell(uwt_\lambda w^{-1}) = \ell(uwt_\lambda) + \ell(w^{-1})$ and $\ell(w^{-1}) = \ell(w)$ so the claim follows. $\square$

The following result can be found in [3, Lemma 4.3] and [20, Lemma 3.2].

Lemma 3.5. *For any positive root $\alpha \in R^+$, we have $\ell(r_\alpha) \leq \langle \alpha^\vee, 2\rho \rangle - 1$. In the case of a simple laced root system, equality always holds.*

4. The superregular affine Bruhat order

We call an element $\lambda \in Q^\vee$ *superregular* if $|\langle \lambda \,,\, \alpha \rangle| \gg 0$ for every $\alpha \in R^+$.[2] In particular, superregular elements are regular. We say that $x = wt_\lambda \in W_{\mathrm{af}}$ is superregular if λ is. We fix once and for all a set of superregular elements $W_{\mathrm{af}}^{\mathrm{sreg}} \subset W_{\mathrm{af}}$.

In the rest of the paper we will say a property, or result holds for "sufficiently superregular" elements $W_{\mathrm{af}}^{\mathrm{ssreg}} \subset W_{\mathrm{af}}$ if there is a positive constant $k \in \mathbb{Z}$ such that the property, or result holds for all $x \in W_{\mathrm{af}}^{\mathrm{sreg}}$ satisfying

$$\text{if } y \in W_{\mathrm{af}} \text{ satisfies } y < x \text{ and } \ell(x) - \ell(y) < k \text{ then } y \in W_{\mathrm{af}}^{\mathrm{sreg}}.$$

We will in general not specify the constant k explicitly but the computation of k will in every case be trivial. The notation $W_{\mathrm{af}}^{\mathrm{ssreg}}$ will thus depend on context.

We say that $x = wt_{v\lambda} \in W_{\mathrm{af}}$ is in the *v-chamber* if λ is regular antidominant. We will say that x and x' are in the same chamber if they are both in the v-chamber for some $v \in W$.

Proposition 4.1. *Let $\lambda \in \tilde{Q}$ be antidominant and superregular and let $x = wt_{v\lambda}$. Then $y = xr_{v\alpha+n\delta} \lessdot x$ if and only if one of the following conditions holds:*

1. *$\ell(wv) = \ell(wvr_\alpha) - 1$ and $n = \langle \lambda \,,\, \alpha \rangle$, giving $y = wr_{v\alpha}t_{v\lambda}$.*
2. *$\ell(wv) = \ell(wvr_\alpha) + \langle \alpha^\vee \,,\, 2\rho \rangle - 1$ and $n = \langle \lambda \,,\, \alpha \rangle + 1$ giving $y = wr_{v\alpha}t_{v(\lambda+\alpha^\vee)}$.*
3. *$\ell(v) = \ell(vr_\alpha) + 1$ and $n = 0$, giving $y = wr_{v\alpha}t_{vr_\alpha\cdot\lambda}$*
4. *$\ell(v) = \ell(vr_\alpha) - \langle \alpha^\vee \,,\, 2\rho \rangle + 1$ and $n = -1$, giving $y = wr_{v\alpha}t_{vr_\alpha(\lambda+\alpha^\vee)}$.*

Proof. Suppose $y = xr_{v\alpha+n\delta} \lessdot x$. For $n \in \mathbb{Z}$, define $f(n) := \ell(t_{v(\lambda+n\alpha^\vee)})$. By Lemma 3.1, we have $f(n) = \sum_{\beta \in R^+} |\langle \lambda + n\alpha^\vee, v^{-1} \cdot \beta \rangle|$ which is a convex function of n. By superregularity of λ, we have

$$f(n) = f(0) - n\langle \alpha^\vee, 2\rho \rangle \tag{3}$$

for sufficiently small values of n. Also we have

$$f(-\langle \lambda, \alpha \rangle) = \ell(t_{vr_\alpha\cdot\lambda}) = f(0)$$

[2]For our purposes $|\langle \lambda \,,\, \alpha \rangle| > 2|W| + 2$ is sufficient.

and thus by superregularity

$$f(-\langle\lambda,\alpha\rangle - n) = f(0) - n\langle\alpha^\vee, 2\rho\rangle$$

for sufficiently small values of n. By convexity we conclude that if n is not close to either 0 or $-\langle\lambda,\alpha\rangle$ then $f(n)$ is not close to $f(0)$. Now write

$$y = wt_{v\lambda}r_{v\alpha}t_{nv\alpha^\vee} = wr_{v\alpha}t_{v(\lambda+(n-\langle\lambda\,,\,\alpha\rangle)\alpha^\vee)}.$$

Since $|\ell(wr_{v\alpha}t_{v(\lambda+(n-\langle\lambda\,,\,\alpha\rangle)\alpha^\vee)}) - \ell(t_{v(\lambda+(n-\langle\lambda\,,\,\alpha\rangle)\alpha^\vee)})| \le |W|$ by superregularity and convexity we may thus assume that either (a) $\lambda + (n - \langle\lambda\,,\,\alpha\rangle)\alpha^\vee$ is antidominant, or (b) $\lambda - n\alpha^\vee$ is antidominant. In case (a), using Lemma 3.4

$$\begin{aligned}\ell(y) &= \ell(wvr_\alpha t_{\lambda+(n-\langle\lambda\,,\,\alpha\rangle)\alpha^\vee}v^{-1}) \\ &= -\ell(wvr_\alpha) + \ell(v^{-1}) + \ell(t_\lambda) + (n - \langle\lambda\,,\,\alpha\rangle)\langle\alpha^\vee\,,\,2\rho\rangle \\ &= \ell(x) + \ell(wv) - \ell(wvr_\alpha) + (n - \langle\lambda\,,\,\alpha\rangle)\langle\alpha^\vee\,,\,2\rho\rangle.\end{aligned}$$

Using Lemma 3.5, we deduce that $n = \langle\lambda\,,\,\alpha\rangle$ or $n = \langle\lambda\,,\,\alpha\rangle + 1$ giving cases (1) and (2) of the Lemma. Similarly, in case (b), we obtain cases (3) and (4) of the Lemma. □

Fix a sufficiently superregular antidominant element $\lambda \in \tilde{Q}$. Let G_λ denote the graph obtained from the restriction of the Hasse diagram of the Bruhat order on W_{af} to the superregular elements $x \in W_{\mathrm{af}}^{\mathrm{sreg}}$ such that $x \le t_{w\lambda}$ for some $w \in W$. We will further direct the edges of G_λ downwards (in the direction of smaller length), so that the $|W|$ vertices $x = t_{v\lambda}$ are the source vertices. By Lemma 4.1 the edges of G_λ either stay within the same chamber (cases (1) and (2)) or go between different chambers (cases (3) and (4)). We call the first kind of edge (or cover) *near* and denote such a cover by $y \lessdot_n x$ and call the second kind *far*, denoting them by $y \lessdot_f x$. By definition the graph obtained from G_λ by keeping only the near edges is a union of the connected components G_λ^v which contain $t_{v\lambda}$, for $v \in W$.

The following combinatorial result makes explicit the relationship between the quantum Bruhat graph and the superregular affine Bruhat order.

Corollary 4.2. *Suppose $\lambda \in \tilde{Q}$ is sufficiently superregular. Each edge $wt_{v\lambda} \to wr_{v\alpha}t_{v\lambda}$ (or $wt_{v\lambda} \to wr_{v\alpha}t_{v(\lambda+\alpha^\vee)}$) in G_λ^v is canonically associated to the edge $wv \to wvr_\alpha$ in $D(W)$. Thus each sufficiently short path $\mathcal{P}$ in $D(W)$ from v to w induces a unique path $\mathcal{Q}$ in G_λ^w, which goes from $t_{v\lambda}$ to $wv^{-1}t_{v\mu}$ where μ equals λ plus the sum of $\alpha^\vee$ over all edges in $\mathcal{Q}$ which are of type (2) (as in Proposition 4.1).*

Proof. The result follows from comparing the definition of the quantum Bruhat graph with cases (1) and (2) of Proposition 4.1. □

We use the phrase "sufficiently short" in Corollary 4.2 since a very long path $\mathcal{P}$ in $D(W)$ will give rise to a path $\mathcal{Q}$ which leaves G_λ, that is, uses non-superregular elements.

Remark 4.1.
(1) In all cases of Proposition 4.1, the positive affine root for the reflection $r_{v\alpha+n\delta}$ is given by $-v\alpha - n\delta$.
(2) Every superregular element has a unique factorization $wt_\lambda v^{-1}$ where $v, w \in W$ and λ is antidominant superregular. In passing to a Bruhat cocover of $wt_\lambda v^{-1}$, λ either stays the same or is replaced by $\lambda+\alpha^\vee$; in the "near" case w is replaced by wr_α with associated quantum Bruhat edge $w \to wr_\alpha$, while

in the "far" case v is replaced by vr_α, with associated quantum Bruhat edge $vr_\alpha \to v$.

Given a (sufficiently short) path $\mathcal{P} \in D(W)$ beginning at $w \in W$, we denote by $x_{\mathcal{P}} \in W_{\mathrm{af}}$ the endpoint of the path in G_λ^w associated to $\mathcal{P}$ via Corollary 4.2. The following Lemma is a translation of [23, Lemma 1] into our language.

Lemma 4.3. *Suppose $\mathcal{P}$ and $\mathcal{P}'$ are two paths in $D(W)$ from w to v of shortest length. Then $x_{\mathcal{P}} = x_{\mathcal{P}'}$.*

Theorem 4.4. *Each tilted Bruhat order $D_u(W)$ is dual to an induced suborder of affine Bruhat order.*

Proof. Let $x(u,w) \in W_{\mathrm{af}}$ be the vertex of G_λ^u (with λ sufficiently superregular) satisfying $x(u,w) = x_{\mathcal{P}}$ for a shortest path $\mathcal{P}$ from u to w in $D(W)$. By Lemma 4.3, $x(u,w)$ does not depend on the choice of $\mathcal{P}$. By Proposition 4.1, the partial order $D_u(W)$ is canonically isomorphic via the map $w \mapsto x(u,w)$ to the dual of the affine Bruhat order restricted to elements $\{x(u,w) \in W_{\mathrm{af}} \mid w \in W\}$. □

5. Affine Bruhat operators

For $X \subset W_{\mathrm{af}}$ let $S[X] = \bigoplus_{x\in X} Sx$ be the free left S-module with basis X. For each $\mu \in P$ and $x = wt_{v\lambda} \in W_{\mathrm{af}}^{\mathrm{ssreg}}$, the *near equivariant affine Bruhat operator* is the left S-module homomorphism $B^\mu : S[W_{\mathrm{af}}^{\mathrm{ssreg}}] \to S[W_{\mathrm{af}}^{\mathrm{sreg}}]$ defined by

$$B^\mu(x) = (\mu - wv\cdot\mu)\, x + \sum_{\alpha\in R^+} \sum_{xr_{v\alpha+n\delta} \lessdot_n x} \langle \alpha^\vee \,,\, \mu\rangle\, xr_{v\alpha+n\delta}$$

Fix a superregular antidominant element $\lambda \in \tilde{Q}$. We call an element σ of $QH^T(G/B)$ *λ-small* if all powers q_μ which occur in σ satisfy the property that $\mu+\lambda$ is superregular antidominant. For each $w \in W$, define the left S-module homomorphism Θ_w^λ from the λ-small elements of $QH^T(G/B)$ to $S[G_\lambda]$ by

$$\Theta_w^\lambda(q_\mu\, \sigma^v) = vw^{-1} t_{w(\lambda+\mu)} = vt_\mu(t_\lambda w^{-1}).$$

The equivariant affine Bruhat operator is related to the equivariant quantum Chevalley formula via the following result.

Proposition 5.1. *Let $\lambda \in \tilde{Q}$ be superregular, $\mu \in P$, $\sigma \in QH^T(G/B)$ be λ-small, and $w \in W$. Then*

$$\Theta_w^\lambda(\sigma * [\mu]) = B^\mu(\Theta_w^\lambda(\sigma))$$

whenever $\Theta_w^\lambda(\sigma)$ is in the domain of B^μ.

Proof. By linearity it suffices to prove the statement for $\sigma = q_\mu \sigma_v$. We have

$$\begin{aligned}
& \Theta_w^\lambda(q_\mu\, \sigma_v * [\mu]) \\
= \;& \Theta_w^\lambda(q_\mu((\mu - v\cdot\mu)\sigma^v + \sum_\alpha \langle\alpha^\vee,\mu\rangle\, \sigma^{vr_\alpha} + \sum_\alpha \langle\alpha^\vee,\mu\rangle\, q_{\alpha^\vee}\sigma^{vr_\alpha})) \\
= \;& (\mu - v\cdot\mu) vw^{-1} t_{w(\lambda+\mu)} \\
+ \;& \sum_\alpha \langle\alpha^\vee,\mu\rangle vr_\alpha w^{-1} t_{w(\lambda+\mu)} + \sum_\alpha \langle\alpha^\vee,\mu\rangle vr_\alpha w^{-1} t_{w(\lambda+\mu+\alpha^\vee)} \\
= \;& B^\mu(vw^{-1}t_{w(\lambda+\mu)}).
\end{aligned}$$

We have used Theorem 2.1, Proposition 4.1, together with the calculation $vr_\alpha w^{-1} = vw^{-1} r_{w\alpha}$. The summations in the equations are as in Theorem 2.1. □

Proposition 5.2. *Let $\mu, \nu \in P$. Then the operators B^μ and B^ν commute as operators on $S[W_{\mathrm{af}}^{\mathrm{ssreg}}]$ (whenever they are defined).*

Proof. Any element $x = wt_{v\lambda} \in W_{\mathrm{af}}^{\mathrm{sreg}}$ is in the image of Θ_v^λ. The result follows immediately from Proposition 5.1, since by the commutativity of $QH^T(G/B)$ one has $\sigma \cdot [\mu] \cdot [\nu] = \sigma \cdot [\nu] \cdot [\mu]$. $\square$

Let $x = wt_{v\lambda}$. The *far equivariant affine Bruhat operator* is the left S-module homomorphism $C^\mu : S[W_{\mathrm{af}}^{\mathrm{ssreg}}] \to S[W_{\mathrm{af}}^{\mathrm{sreg}}]$ defined by

$$C^\mu(x) = (\mu - v \cdot \mu)\, x + \sum_{\alpha \in R^+} \sum_{xr_{v\alpha+n\delta} \lessdot_f x} \langle \alpha^\vee \,,\, \mu \rangle \, xr_{v\alpha+n\delta}.$$

The operators C^μ and B^μ are related by the following formula when acting on the special element $\sum_{w \in W} t_{w\lambda}$.

Lemma 5.3. *Let $\lambda \in \tilde{Q}$ be a sufficiently superregular antidominant coweight and $\mu^1, \mu^2, \ldots, \mu^k \in P$ be a sequence of integral weights. Then*

$$C^\mu \left(B^{\mu^k} \cdots B^{\mu^2} B^{\mu^1} \cdot \sum_{w \in W} t_{w\lambda} \right) = B^{\mu^k} \cdots B^{\mu^2} B^{\mu^1} \cdot \left(B^\mu \cdot \sum_{w \in W} t_{w\lambda} \right). \tag{4}$$

Proof. A term of $B^{\mu^k} \cdots B^{\mu^2} B^{\mu^1} \cdot t_{w\lambda}$ is indexed by a multipath (a path allowed to stay at a vertex for multiple steps)

$$\mathcal{P} = \{ w = w^{(0)} \to w^{(1)} \to w^{(2)} \to \cdots \to w^{(k)} \}$$

in $D(W)$, where for each $i \in [1, k]$, we have (i) $w^{(i)} = w^{(i-1)}$ or (ii) $w^{(i)} = w^{(i-1)} r_{\alpha^{(i)}}$. Each such path $\mathcal{P}$ contributes a term $a_\mathcal{P}\, x_\mathcal{P}$, where $a_\mathcal{P} = \prod_i a_i$ with $a_i = \mu^{(i)} - w^{(i)} \cdot \mu^{(i)}$ in case (i) and $a_i = \langle (\alpha^{(i)})^\vee \,,\, \mu \rangle$ in case (ii). The left hand side of (4) can thus be given as the sum over pairs $(\mathcal{P}, \mathcal{Q})$ where $\mathcal{P}$ is a multipath from w to v in $D(W)$ of length k, and $\mathcal{Q}$ is a multipath from u to w of length 1. If $x_\mathcal{P} = vw^{-1} t_{w\mu}$ then $(\mathcal{P}, \mathcal{Q})$ contributes $a_{\mathcal{P},\mathcal{Q}} x_{\mathcal{P},\mathcal{Q}}$ where $x_{\mathcal{P},\mathcal{Q}} = vu^{-1} t_{u\mu'}$ with $\mu' = \mu$ or $\mu' = \mu + \alpha^\vee$ for some $\alpha \in R^+$. The coefficient $a_{\mathcal{P},\mathcal{Q}}$ is equal to $a_\mathcal{P}\, a_\mathcal{Q}$ where $a_\mathcal{Q} = \mu - w \cdot \mu$ if $u = w$ and $a_\mathcal{Q} = \langle \alpha^\vee \,,\, \mu \rangle$ if $u = wr_\alpha$.

To obtain (4) we send the pair $(\mathcal{P}, \mathcal{Q})$ to the multipath

$$\mathcal{P}' = \{ u \to w = w^{(0)} \to w^{(1)} \to w^{(2)} \to \cdots \to w^{(k)} \}$$

and we observe that $x_{\mathcal{P}'} = x_{\mathcal{P},\mathcal{Q}}$ and $a_{\mathcal{P}'} = a_{\mathcal{P},\mathcal{Q}}$, where $\mathcal{P}'$ is weighted according to the sequence $\mu, \mu^{(1)}, \ldots, \mu^{(k)}$. Note that in the case that $u = wr_\alpha$, the first step of $\mathcal{P}'$ corresponds to a cover $xr_{(wr_\alpha)\alpha+n\delta} \lessdot x$ where $x = t_{wr_\alpha \lambda}$. $\square$

6. Homology of affine Grassmannian

6.1. **Affine nilHecke ring.** Let $\mathbb{A}_{\mathrm{af}}$ denote the *affine nilHecke ring* of Kostant and Kumar. Our conventions here differ slightly from those in [13] but agree with those in [14], and we refer to the latter for a discussion of the differences. We use the action of W_{af} on P induced by the action (1), under which translation elements act trivially, or equivalently, r_0 acts by r_θ. $\mathbb{A}_{\mathrm{af}}$ is the ring with a 1 given by generators

$\{A_i \mid i \in I_{\mathrm{af}}\} \cup \{\lambda \mid \lambda \in P\}$ and the relations

$$\begin{aligned} A_i\,\lambda &= (r_i \cdot \lambda)\,A_i + \langle \lambda, \alpha_i^\vee \rangle \cdot 1 && \text{for } \lambda \in P, \\ A_i\,A_i &= 0, \\ \underbrace{A_i A_j A_i \cdots}_{m} &= \underbrace{A_j A_i A_j \cdots}_{m} && \text{if } \underbrace{r_i r_j r_i \cdots}_{m} = \underbrace{r_j r_i r_j \cdots}_{m}, \end{aligned}$$

where the "scalars" $\lambda \in P$ commute with other scalars. Let $w \in W_{\mathrm{af}}$ and let $w = r_{i_1} \cdots r_{i_l}$ be a reduced decomposition of w. Then $A_w := A_{i_1} \cdots A_{i_l}$ is a well defined element of $\mathbb{A}_{\mathrm{af}}$, where $A_{\mathrm{id}} = 1$. $\mathbb{A}_{\mathrm{af}}$ is a free left S-module (and a free right S-module) with basis $\{A_w \mid w \in W_{\mathrm{af}}\}$. Note that we have

$$A_x A_y = \begin{cases} A_{xy} & \text{if } \ell(x) + \ell(y) = \ell(xy), \\ 0 & \text{otherwise.} \end{cases}$$

We have the following commutation relation which can be established by induction; see [13].

Lemma 6.1. *For $x \in W_{\mathrm{af}}$ and $\lambda \in P$,*

$$A_x \lambda = (x \cdot \lambda) A_x + \sum_{\substack{\beta \in R_{\mathrm{af}}^{\mathrm{re}+} \\ x r_\beta \lessdot x}} \langle \beta^\vee, \lambda \rangle A_{x r_\beta}.$$

6.2. **Equivariant homology of affine Grassmannian.** The affine Grassmannian Gr_G associated to G is the ind-scheme $G(\mathcal{K})/G(\mathcal{O})$ where $\mathcal{K} = \mathbb{C}((t))$ denotes the ring of formal Laurent series and $\mathcal{O} = \mathbb{C}[[t]]$ is the ring of formal power series. The space Gr_G is weakly homotopy equivalent to the space ΩK of based loops into the maximal compact subgroup $K \subset G$ and thus the homology $H_*(\mathrm{Gr}_G)$ and equivariant homology $H_T(\mathrm{Gr}_G)$ inherits a ring structure via Pontryagin multiplication.

The ring $H_T(\mathrm{Gr}_G)$ is a free $S = H_T(\mathrm{pt})$-module with basis given by the T-equivariant Schubert classes $\{\xi_x \mid x \in W_{\mathrm{af}}^-\}$. The affine nilHecke ring $\mathbb{A}_{\mathrm{af}}$ acts on $H_T(\mathrm{Gr}_G)$ by

$$A_y \cdot \xi_z = \begin{cases} \xi_{yz} & \text{if } \ell(yz) = \ell(y) + \ell(z) \text{ and } yz \in W_{\mathrm{af}}^-, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

and $S \subset \mathbb{A}_{\mathrm{af}}$ acts via the usual S-module structure of $H_T(\mathrm{Gr}_G)$.

We now describe Peterson's model for $H_T(\mathrm{Gr}_G)$ [22]. We refer the reader to [14] for more details.

Let $Z_{\mathbb{A}_{\mathrm{af}}}(S) \subset \mathbb{A}_{\mathrm{af}}$ denote the centralizer of S in $\mathbb{A}_{\mathrm{af}}$, called the *Peterson subalgebra* in [14]. Let $J \subset \mathbb{A}_{\mathrm{af}}$ denote the left ideal

$$J = \sum_{w \in W \setminus \{\mathrm{id}\}} \mathbb{A}_{\mathrm{af}} A_w.$$

The following two theorems are due to Peterson [22]. We refer the reader to [14, Lemma 3.3 and Theorem 4.4] for a proof of Theorem 6.2.

Theorem 6.2. *There is an S-algebra isomorphism $j : H_T(\mathrm{Gr}_G) \to Z_{\mathbb{A}_{\mathrm{af}}}(S)$ such that*

$$j(\xi_x) = A_x \mod J \quad \text{and} \quad j(\xi) \cdot \xi' = \xi\,\xi'$$

for $\xi, \xi' \in H_T(\mathrm{Gr}_G)$. The element $j(\xi_x)$ is determined by the properties: (1) $j(\xi_x) \in Z_{\mathbb{A}}(S)$ and (2) $j(\xi_x) = A_x \mod J$.

Define $j_x^y \in S$ by

$$j(\xi_x) = \sum_y j_x^y A_y.$$

The elements $j_x^y \in S$ are polynomials of degree $\ell(y) - \ell(x)$ in the simple roots $\{\alpha_i \mid i \in I\}$.

Theorem 6.3. *For $x, z \in W_{\mathrm{af}}^-$ we have*

$$\xi_x \, \xi_z = \sum_y j_x^y \, \xi_{yz}$$

where the summation is over $y \in W_{\mathrm{af}}$ such that $yz \in W_{\mathrm{af}}^-$ and $\ell(yz) = \ell(y) + \ell(z)$.

Proof. By Theorem 6.2 we have

$$j(\xi_x) \cdot \xi_z = \xi_x \, \xi_z \tag{6}$$

where the action is as in (5). The statement then follows from the observation that in a length-additive product $yz \notin W_{\mathrm{af}}^-$ if $z \notin W_{\mathrm{af}}^-$. □

7. Generating elements of the Peterson subalgebra

We now describe a method for producing elements of the Peterson subalgebra. Define the left S-module isomorphism $\Upsilon : S[W_{\mathrm{af}}] \to \mathbb{A}_{\mathrm{af}}$ by

$$\Upsilon\Big(\sum_{x \in W_{\mathrm{af}}} a_x \, x\Big) = \sum_{x \in W_{\mathrm{af}}} a_x \, A_x$$

for $a_x \in S$. Let $x = wt_{v\lambda}$. For $\mu \in P$, the *twisted equivariant affine Bruhat operators* are the left S-module homomorphisms $\tilde{B}^\mu, \tilde{C}^\mu : S[W_{\mathrm{af}}^{\mathrm{ssreg}}] \to S[W_{\mathrm{af}}^{\mathrm{sreg}}]$ defined by

$$\tilde{B}^\mu(x) = (v^{-1}\mu - w\mu)x + \sum_{\alpha \in R^+} \sum_{xr_{v\alpha+n\delta} \lessdot_n x} \langle v\alpha^\vee \, , \, \mu\rangle \, xr_{v\alpha+n\delta}$$

and

$$\tilde{C}^\mu(x) = (v^{-1}\mu - \mu)x - \sum_{\alpha \in R^+} \sum_{xr_{v\alpha+n\delta} \lessdot_f x} \langle v\alpha^\vee \, , \, \mu\rangle \, xr_{v\alpha+n\delta}.$$

Lemma 7.1. *Let $f \in S[W_{\mathrm{af}}^{\mathrm{ssreg}}]$. Then $\Upsilon(f) \in Z_{\mathbb{A}_{\mathrm{af}}}(S)$ if and only if for each $\mu \in P$ we have*

$$\tilde{B}^\mu(f) = \tilde{C}^\mu(f).$$

Proof. The statement follows immediately from Lemma 6.1, Proposition 4.1, and Remark 4.1 (1). (If we literally apply Lemma 6.1, the terms $(v^{-1}\mu - w\mu)x$ in $\tilde{B}^\mu(x)$ and $(v^{-1}\mu - \mu)x$ in $\tilde{C}^\mu(x)$ would need to be negated; since x does not occur elsewhere in the formula, the stated claim is still true.) □

Theorem 7.2. *Let λ be a sufficiently superregular antidominant coweight and $\mu^1, \mu^2, \ldots, \mu^k \in P$ be a sequence of integral weights. Then the element*

$$\Upsilon\Big(B^{\mu^k} \cdots B^{\mu^2} B^{\mu^1} \cdot \sum_{w \in W} t_{w\lambda}\Big)$$

lies in the Peterson subalgebra $Z_{\mathbb{A}_{\mathrm{af}}}(S)$.

Proof. By Lemma 7.1, it suffices to check that

$$f = B^{\mu^k} \cdots B^{\mu^2} B^{\mu^1} \cdot \sum_{w \in W} t_{w\lambda}$$

satisfies $\tilde{B}^\mu(f) = \tilde{C}^\mu(f)$. The coefficient of $x = wt_{v\lambda}$ in $\tilde{B}^\mu(f)$ (resp. $\tilde{C}^\mu(f)$) is equal to the coefficient of x in $B^{v^{-1}\cdot\mu}(f)$ (resp. $C^{v^{-1}\mu}(f)$). Thus it suffices to show that for all $\mu \in P$ we have $B^\mu(f) = C^\mu(f)$. But using Proposition 5.2 this is exactly the statement of Lemma 5.3. □

8. Formulae for affine Schubert classes

For $w \in W$, let us say that a polynomial

$$\mathfrak{S}_w = \sum_{i_1, i_2, \ldots, i_k} a_{i_1,\ldots,i_k} q_{\lambda(i_1,\ldots,i_k)} \otimes \omega_{i_1}\omega_{i_2}\cdots\omega_{i_k} \in S[q] \otimes_{\mathbb{Z}} \mathbb{Z}[\omega_i \mid i \in I],$$

where $a_{i_1,\ldots,i_k} \in S$ and $\lambda(i_1,\ldots,i_k) \in \oplus_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i^\vee$ is an *equivariant quantum Schubert polynomial* if its image in $QH^T(G/B)$ (obtained by replacing ω_i with $[\omega_i]$) equals the quantum Schubert class σ^w. There are many choices for such a polynomial.

Let us write $b(\lambda; \mu^1, \mu^2, \ldots, \mu^k) \in Z_{\mathbb{A}_{\mathrm{af}}}(S)$ for the element described by Theorem 7.2. The following formula writes affine Schubert classes in terms of quantum Schubert classes.

Theorem 8.1. *Let $\mathfrak{S}_w \in S[q] \otimes_{\mathbb{Z}} \mathbb{Z}[\omega_i \mid i \in I]$ as above be an equivariant quantum Schubert polynomial representing the class $\sigma^w \in QH^T(G/B)$, and let $\lambda \in \tilde{Q}$ be sufficiently superregular. Then*

$$j(\xi_{wt_\lambda}) = \sum_{i_1,i_2,\ldots,i_k} a_{i_1,\ldots,i_k} b(\lambda + \lambda(i_1,\ldots,i_k); \omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_k}). \tag{7}$$

Proof. Let a denote the expression on the right hand side of (7). By Theorem 7.2, $a \in Z_{\mathbb{A}_{\mathrm{af}}}(S)$. By Theorem 6.2, it suffices to show that a contains a unique Grassmannian term A_{wt_λ} with coefficient 1. Let $\underline{i} = (i_1, \ldots, i_k)$. We have

$$\begin{aligned} a &= \Upsilon\Big(\sum_{\underline{i}} a_{\underline{i}} B^{\omega_{i_k}} \cdots B^{\omega_{i_1}} \sum_{w\in W} t_{w(\lambda+\lambda(\underline{i}))}\Big) \\ &= \Upsilon\Big(\sum_{\underline{i}} a_{\underline{i}} B^{\omega_{i_k}} \cdots B^{\omega_{i_1}} \big(t_{\lambda+\lambda(\underline{i})} + \sum_{w\in W\setminus\{\mathrm{id}\}} t_{w(\lambda+\lambda(\underline{i}))}\big)\Big) \end{aligned}$$

By Lemma 3.3 and the fact that λ is sufficiently superregular, it is clear that Grassmannian terms cannot come from any term with $w \neq \mathrm{id}$. By Proposition 5.1 (applied with $w = \mathrm{id}$ and $\sigma = a_{\underline{i}} q_{\lambda(\underline{i})}$), we have

$$\begin{aligned} &\Upsilon\Big(\sum_{\underline{i}} a_{\underline{i}} B^{\omega_{i_k}} \cdots B^{\omega_{i_1}} t_{\lambda+\lambda(\underline{i})}\Big) \\ &= \Upsilon\Big(\sum_{\underline{i}} a_{\underline{i}} B^{\omega_{i_k}} \cdots B^{\omega_{i_1}} \Theta^\lambda_{\mathrm{id}}(q_{\lambda(\underline{i})})\Big) \\ &= \Upsilon\Big(\Theta^\lambda_{\mathrm{id}}\Big(\sum_{\underline{i}} a_{\underline{i}} q_{\lambda(\underline{i})} * [\omega_{i_1}] * \cdots * [\omega_{i_k}]\Big)\Big) \\ &= \Upsilon(\Theta^\lambda_{\mathrm{id}}(\sigma^w)) = A_{wt_\lambda}. \end{aligned}$$

where we have used our assumption that $\mathfrak{S}_w$ represents the class σ^w. □

Remark 8.1.
(1) In Theorem 8.1 (and many other places in the paper) it is possible to use the operators C^μ instead of B^μ to obtain similar results.
(2) Theorem 8.1 can be evaluated at 0 via $\phi_0 : S \to \mathbb{Z}$ to give a formula for $\phi_0(j(\xi_{wt_\lambda}))$ in terms of non-equivariant quantum Schubert polynomial. The elements $\phi_0(j(\xi_{wt_\lambda}))$ lie inside what is called the *affine Fomin-Stanley subalgebra* in [14], and are related to the theory of affine Stanley symmetric functions. See [7, 20] for discussions on how to produce (non-equivariant) quantum Schubert polynomials.

Let us call $a = \sum_{x \in W_{\mathrm{af}}} a_x A_x \in \mathbb{A}_{\mathrm{af}}$ *superregular* if $a_x = 0$ for all $x \in W_{\mathrm{af}} \setminus W_{\mathrm{af}}^{\mathrm{sreg}}$.

Corollary 8.2. *The elements $b(\lambda; \mu^1, \mu^2, \ldots, \mu^k)$ span the set of superregular elements of $Z_{\mathbb{A}_{\mathrm{af}}}(S)$.*

Proof. Let a be a superregular element in $Z_{\mathbb{A}_{\mathrm{af}}}(S)$. By Theorem 6.2, it is an S-linear combination of $j(\xi_x)$ where x is superregular. By Theorem 8.1, $j(\xi_x)$ lies in the span of the elements $b(\lambda; \mu^1, \mu^2, \ldots, \mu^k)$. □

Corollary 8.3. *Let $\mu \in P$ be an integral weight and $\Upsilon(f) \in Z_{\mathbb{A}_{\mathrm{af}}}(S)$ for a sufficiently superregular f. Then $\Upsilon(B^\mu(f)) \in Z_{\mathbb{A}_{\mathrm{af}}}(S)$.*

Proof. Follows immediately from Corollary 8.2 and Theorem 7.2. □

The following Proposition is contained in [14, Proposition 4.5].

Proposition 8.4. *Let $\lambda \in \tilde{Q}$. Then $j(\xi_{t_\lambda}) = \sum_{\mu \in W \cdot \lambda} A_{t_\mu}$.*

The superregular case of Proposition 8.4 follows from Theorem 8.1. The general case can be obtained by a direct calculation, similar to (but simpler than) Proposition 4.1.

Proposition 8.5. *Let λ be superregular antidominant. Then*

$$j(\xi_{r_i t_\lambda}) = b(\lambda; \omega_i) = \Upsilon(B^{\omega_i} \sum_{w \in W} t_{w\lambda})$$
$$= \sum_{w \in W} \left((\omega_i - w\omega_i) A_{t_{w\lambda}} + \sum_{\alpha \in R^+} \langle \alpha^\vee, \omega_i \rangle A_{r_{w\alpha} t_{w\lambda}} + \sum_{\alpha \in R^+} \langle \alpha^\vee, \omega_i \rangle A_{r_{w\alpha} t_{w(\lambda+\alpha^\vee)}} \right)$$

where ω_i denotes the i-th fundamental weight, and the two inner summations are as in Theorem 2.1.

Proof. This follows immediately from Theorem 8.1 and the fact that $\sigma^{r_i} = [\omega_i]$ in $QH^T(G/B)$. □

9. Borel case

Proposition 9.1. *Let $x \in W_{\mathrm{af}}^-$ and $\lambda \in \tilde{Q}$. Then*

$$\xi_x \, \xi_{t_\lambda} = \xi_{x t_\lambda}.$$

Proof. By Lemma 3.4, $\ell(x) + \ell(t_\lambda) = \ell(x t_\lambda)$. The proposition follows immediately from Proposition 8.4 and Theorem 6.3. □

In particular $\{\xi_{t_\lambda} \mid \lambda \in \tilde{Q}\}$ is a multiplicatively closed set that contains no zero divisors. So it makes sense to consider $H_T^t(\mathrm{Gr}_G) = H_T(\mathrm{Gr}_G)[\xi_t^{-1} \mid t \in \tilde{Q}]$. Let us

also define $QH_q^T(G/B) = QH^T(G/B)[q_i^{-1} \mid i \in I]$ and for $\alpha^\vee = \sum_i a_i \alpha_i^\vee \in Q^\vee$ we write $q_{\alpha^\vee} := \prod_{i \in I} q_i^{a_i}$.

Let $\psi : H_T^t(\mathrm{Gr}_G) \to QH_q^T(G/B)$ be the S-module homomorphism defined by

$$\xi_{wt_\lambda}\, \xi_{t_\mu}^{-1} \mapsto q_{\lambda-\mu}\sigma^w.$$

This map is well-defined by Proposition 9.1 and is clearly an S-module isomorphism. Our main theorem is the following.

Theorem 9.2. *The map $\psi : H_T^t(\mathrm{Gr}_G) \to QH_q^T(G/B)$ is an S-algebra isomorphism.*

Proof. It is enough to show that $H_T^t(\mathrm{Gr}_G)$ satisfies the ψ-preimage of the quantum equivariant Chevalley formula (Theorem 2.1) since this completely determines $QH^T(G/B)$ (see [21]). By Proposition 9.1, it is enough to calculate the product $\xi_{r_i t_\lambda}\, \xi_{wt_\mu}$ in $H_T(\mathrm{Gr}_G)$ for superregular antidominant $\lambda, \mu \in \tilde{Q}$. One does so using Proposition 8.5 and Theorem 6.3. For each term $A_{r_{w\alpha} t_{w\lambda}}$ in $j(r_i t_\lambda)$ one obtains a term $\xi_{wr_\alpha t_{\lambda+\mu}}$ in the product $\xi_{r_i t_\lambda}\, \xi_{wt_\mu}$ since

$$(r_{w\alpha} t_{w\lambda})\;(wt_\mu) = wr_\alpha w^{-1} wt_\lambda w^{-1} wt_\mu = wr_\alpha t_{\lambda+\mu}$$

is a length-additive product and $wr_\alpha t_{\lambda+\mu} \in W_{\mathrm{af}}^-$ by Lemma 3.3. The analogous statement holds for terms of the form $r_{w\alpha} t_{w(\lambda+\alpha^\vee)}$, thus ensuring that the the product $\xi_{r_i t_\lambda}\, \xi_{wt_\mu}$ contains terms of the form $\xi_{wr_\alpha t_{\mu+\lambda+\alpha^\vee}}$ where $\ell(wr_\alpha) = \ell(w) - 2\langle \alpha^\vee\,,\, \rho\rangle + 1$. Furthermore, for $v \neq w$, $(r_{v\alpha} t_{v\lambda})\;(wt_\mu)$ is never a length-additive product since $\ell(t_{w^{-1}v\lambda+\mu}) \ll \ell(t_\lambda) + \ell(t_\mu)$. Similarly $(r_{v\alpha} t_{v(\lambda+\alpha^\vee)})\;(wt_\mu)$ is never a length-additive product. Similar computations hold for the equivariant terms $(\omega_i - w\omega_i)A_{t_{w\lambda}}$ in $j(r_i t_\lambda)$. Thus

$$\xi_{r_i t_\lambda}\, \xi_{wt_\mu} = (\omega_i - w\omega_i)\xi_{wt_{\lambda+\mu}} + \sum_{\alpha \in R^+} \langle \alpha^\vee, \omega_i\rangle \xi_{wr_\alpha t_{\lambda+\mu}} + \sum_{\alpha \in R^+} \langle \alpha^\vee, \omega_i\rangle \xi_{wr_\alpha t_{\mu+\lambda+\alpha^\vee}}$$

where the two summations are exactly as in Theorem 2.1. Applying ψ gives exactly Theorem 2.1 with both sides multiplied by $q_{\lambda+\mu}$. □

The following corollary writes the equivariant Gromov-Witten invariants of G/B in terms of Schubert structure constants of $H_T(\mathrm{Gr}_G)$.

Corollary 9.3. *Let $w, v, u \in W$ and $\lambda \in Q^\vee$. Then the equivariant three point Gromov-Witten invariant $c_{w,v}^{u,\lambda}$ is equal to the coefficient of ξ_z in the product $\xi_x\, \xi_y \in H_T(\mathrm{Gr}_G)$, where $x = wt_\eta, y = vt_\kappa, z = ut_\mu \in W_{\mathrm{af}}^-$ and $\lambda = \mu - \eta - \kappa$.*

Now we can write all the coefficients j_x^y in terms of three point genus zero Gromov-Witten invariants of G/B, and conversely.

Theorem 9.4. *Let $x = wt_\lambda \in W_{\mathrm{af}}^-$ and $y = ut_\nu \in W_{\mathrm{af}}$ where we assume $\nu \in Q^\vee$ is superregular. Let $v \in W$ be the unique element such that $v^{-1}\nu \in \tilde{Q}$. Then*

$$j_x^y = c_{w,v}^{uv, v^{-1}\nu-\lambda},$$

where $c_{w,v}^{u,\kappa} = 0$ if κ is not a nonnegative sum of simple coroots. Conversely, suppose $f, g, h \in W$ and $\eta \in Q^\vee$ are given. Then

$$c_{f,g}^{h,\eta} = j_{ft_\lambda}^{hg^{-1}t_{g(\eta+\lambda)}}$$

for sufficiently superregular antidominant $\lambda \in \tilde{Q}$.

Proof. Let $z = vt_\mu \in W_{\mathrm{af}}^-$ where μ is chosen to be superregular. By Theorem 6.3, we know that j_x^y is the coefficient of $\xi_{yz} = \xi_{uvt_{v^{-1}\nu+\mu}}$ in $\xi_x\xi_z$, as long as $yz \in W_{\mathrm{af}}^-$ and $\ell(yz) = \ell(y) + \ell(z)$. Using Lemmata 3.3 and 3.4, we check that the latter two conditions are immediate with our assumptions. Applying the map ψ of Theorem 9.2, we see that j_x^y is equal to the coefficient of $q_{v^{-1}\nu+\mu}\sigma^{uv}$ in $q_{\mu+\lambda}\sigma^w\sigma^v$. To obtain the second statement from the first, it suffices to note that $\eta + \lambda$ is superregular antidominant if $\lambda \in \tilde{Q}$ is sufficiently superregular. □

Remark 9.1. Theorem 9.4 only writes j_x^y for superregular $y \in W_{\mathrm{af}}$ in terms of Gromov Witten invariants of G/B. To obtain the rest of the j-coefficients, one can use Proposition 8.4 and the observation that for any $y \in W_{\mathrm{af}}$ there is a length additive product yt_μ (with $\mu \in Q^\vee$) which is superregular.

Mihalcea [21] has shown that equivariant Gromov-Witten invariants are polynomials in simple roots with nonnegative coefficients (in fact Mihalcea uses negative simple roots). As a consequence we obtain a positivity result for the j-coefficients, and hence for all affine homology structure constants of $H_T(\mathrm{Gr}_G)$.

Corollary 9.5. *All equivariant homology Schubert structure constants of $H_T(\mathrm{Gr}_G)$ are nonegative polynomials in the simple roots. For each $x \in W_{\mathrm{af}}^-$ and $y \in W_{\mathrm{af}}$, the polynomial $j_x^y \in S$ is a nonnegative polynomial in the simple roots.*

It would be interesting to obtain a direct proof of Corollary 9.5 which does not appeal to quantum cohomology, even for the nonequivariant ($\ell(x) = \ell(y)$) case.

10. Parabolic case

Let $P \subset G$ be a standard parabolic subgroup. Following Peterson, up to localization we show that $QH^T(G/P)$ is a quotient of $H_T(\mathrm{Gr}_G)$.

10.1. **Extended affine Weyl group.** Recall that W acts on the coweight lattice $P^\vee$. Therefore we may define the extended affine Weyl group $\widetilde{W} \cong W \ltimes P^\vee$; as before, the element in $\widetilde{W}$ corresponding to $\lambda \in P^\vee$ is denoted t_λ. $\widetilde{W}$ acts on the affine root lattice Q_{af} by the same formula as (1) with $\lambda \in P^\vee$. There is an induced action of $\widetilde{W}$ on $Q \cong Q_{\mathrm{af}}/\mathbb{Z}\delta$.

$\widetilde{W}$ is not a Coxeter group. However it still permutes $R_{\mathrm{af}}^{\mathrm{re}}$, so for $x \in \widetilde{W}$ we can define its inversion set $\mathrm{Inv}(x)$ and length $\ell(x)$ in the same way as for $x \in W_{\mathrm{af}}$. The set $\widetilde{W}^0 = \{\tau \in \widetilde{W} \mid \ell(\tau) = 0\}$ of elements of $\widetilde{W}$ of length zero, forms a subgroup of $\widetilde{W}$ since it is the stabilizer of the set R_{af}^+.

Let $\delta = \alpha_0 + \theta = \sum_{i \in I_{\mathrm{af}}} a_i\alpha_i$ be the null root. A node $i \in I_{\mathrm{af}}$ is called *special* if $a_i = 1$, or equivalently, if there is an automorphism of the affine Dynkin diagram taking the node i to the Kac 0 node. Denote by $I^s \subset I_{\mathrm{af}}$ the set of special nodes. The nodes in $I^s \setminus \{0\}$ are also called *cominuscule*. The abelian group $\Sigma = P^\vee/Q^\vee$ consists of the elements $\omega_i^\vee + Q^\vee$ for $i \in I^s$ where $\omega_0^\vee = 0$.

There is an isomorphism $\Sigma \cong \widetilde{W}^0$ which can be described as follows. Let $i \in I^s$. Addition by the element $-\omega_i^\vee + Q^\vee \in \Sigma$, defines a permutation of the elements of $P^\vee/Q^\vee$ or equivalently, a permutation of the set I^s. This permutation extends uniquely to an automorphism τ_i of the affine Dynkin diagram and satisfies $\tau_i(i) = 0$. It acts on Q_{af} by $\tau_i(\alpha_j) = \alpha_{\tau_i(j)}$ for all $j \in I_{\mathrm{af}}$. It follows that $\tau_i(\delta) = \delta$ so that $\tau_i \in \widetilde{W}^0$. The above isomorphism is given by $-\omega_i^\vee + Q^\vee \mapsto \tau_i$. Note that τ_0 is the identity in $\widetilde{W}$.

Define $v_i \in W$ to the shortest element such that $v_i\omega_i = w_0\omega_i$ and let $\omega_0 = 0$ so that $v_0 = 1$. Then

$$\tau_i = v_i t_{-\omega_i^\vee}. \tag{8}$$

Moreover $W^s = \{v_i \mid i \in I^s\}$ forms a subgroup of W and the map $\widetilde{W}^0 \to W^s$ given by $\tau_i \mapsto v_i$, is an isomorphism.

10.2. **Affinization of W_P.** Let $L_P \subset G$ be the Levi factor of the parabolic subgroup $P \subset G$. Say that L_P has Dynkin node set I_P, root system R_P, root lattice Q_P, coroot lattice $Q_P^\vee$, coweight lattice $P_P^\vee$, and Weyl group W_P. Let W^P denote the set of minimal length coset representatives in W/W_P. Define

$$(W_P)_{\mathrm{af}} = W_P \ltimes Q_P^\vee = \{wt_\lambda \in W_{\mathrm{af}} \mid w \in W_P, \lambda \in Q_P^\vee\}. \tag{9}$$

L_P has affine root lattice $(Q_P)_{\mathrm{af}} = Q_P \oplus \mathbb{Z}\delta \subset Q_{\mathrm{af}}$, affine Weyl group $(W_P)_{\mathrm{af}}$, and extended affine Weyl group $\widetilde{W_P} = W_P \ltimes P_P^\vee$.

Let $I_P = \bigsqcup_{m=1}^k I_m$ be the partition of the node set of I_P according to the connected components of the subgraph of the Dynkin diagram of G induced by the subset of nodes I_P. Write R_m, $Q_m^\vee$, $P_m^\vee$ for the irreducible subrootsystem, coroot lattice and coweight lattice respectively. Then we have an isomorphism of abelian groups

$$\Sigma_P \cong \prod_{m=1}^k \Sigma_m \tag{10}$$

where $\Sigma_P = P_P^\vee/Q_P^\vee$ and $\Sigma_m = P_m^\vee/Q_m^\vee$.

Let $(I_m)_{\mathrm{af}} = I_m \cup \{0_m\}$; the zero nodes for various m are distinct. Write $(Q_m)_{\mathrm{af}} = Q_m \oplus \mathbb{Z}\delta \subset (Q_P)_{\mathrm{af}} \subset Q_{\mathrm{af}}$. Let $\alpha_{0_m} = \delta - \theta_m \in (Q_m)_{\mathrm{af}}$ where $\theta_m \in R_m^+$ is the highest root. Then $(Q_m)_{\mathrm{af}}$ has basis $\{\alpha_i \mid i \in (I_m)_{\mathrm{af}}\}$. Σ_m acts on $(Q_m)_{\mathrm{af}}$, inducing a permutation of $(I_m)_{\mathrm{af}}$ defined by $\tau(\alpha_i) = \alpha_{\tau(i)}$ for $i \in (I_m)_{\mathrm{af}}$. Note that $\mathbb{Z}\delta \subset (Q_m)_{\mathrm{af}} \subset Q_{\mathrm{af}}$ is fixed under the action of Σ_m.

10.3. **$(W^P)_{\mathrm{af}}$.** Let

$$\begin{aligned}(R_P)_{\mathrm{af}}^+ &= \{\beta \in R_{\mathrm{af}}^+ \mid \bar{\beta} \in R_P\}\\ (W^P)_{\mathrm{af}} &= \{x \in W_{\mathrm{af}} \mid x \cdot \beta > 0 \text{ for all } \beta \in (R_P)_{\mathrm{af}}^+\}.\end{aligned}$$

Remark 10.1. Suppose $P \neq G$, or equivalently, $\theta \notin R_P$. Then $r_0 \in (W^P)_{\mathrm{af}}$, since r_0 has the lone inversion $\alpha_0 = \delta - \theta \notin (R_P)_{\mathrm{af}}^+$.

Lemma 10.1. *$wt_\lambda \in (W^P)_{\mathrm{af}}$ if and only if, for every $\alpha \in R_P^+$, if $w\alpha > 0$ then $\langle \lambda\,,\,\alpha\rangle = 0$ and if $w\alpha < 0$ then $\langle \lambda\,,\,\alpha\rangle = -1$.*

Proof. For any $x \in W_{\mathrm{af}}$ and $\alpha \in R^+$, if $\alpha + n\delta \in \mathrm{Inv}(x)$ for some $n \in \mathbb{Z}_{\geq 0}$ then $\alpha \in \mathrm{Inv}(x)$. Similarly, if $-\alpha + n\delta \in \mathrm{Inv}(x)$ for some $n \in \mathbb{Z}_{>0}$ then $\delta - \alpha \in \mathrm{Inv}(x)$. Therefore $wt_\lambda \in (W^P)_{\mathrm{af}}$ if and only if, for every $\alpha \in R_P^+$, $\alpha \notin \mathrm{Inv}(wt_\lambda)$ and $\delta - \alpha \notin \mathrm{Inv}(wt_\lambda)$. The lemma follows straightforwardly from these conditions. □

Lemma 10.2. *Suppose that $wt_\lambda \in (W^P)_{\mathrm{af}}$, R_P is an irreducible root system, $\langle \lambda\,,\,\alpha_j\rangle \neq 0$ for some $j \in I_P$, and $w = w_1w_2$ where $w_1 \in W^P$ and $w_2 \in W_P$. Then*

(1) *The node j is cominuscule in I_P.*

(2) *For* $\alpha \in R_P^+$,

$$\langle \lambda\,,\,\alpha\rangle = \begin{cases} -1 & \text{if } \alpha_j \text{ occurs in } \alpha \\ 0 & \text{otherwise.} \end{cases}$$

(3) $w_2 = v_j^P$, *with notation as in Section 10.1, with respect to the cominuscule node* j *in* I_P.

Proof. We shall use Lemma 10.1 repeatedly without further mention. We have $\langle \lambda\,,\,\alpha_j\rangle = -1$. Suppose α_j occurs in $\alpha \in R_P^+$, that is, $\alpha = \sum_{i \in I_P} a_i \alpha_i$ with $a_j > 0$ and all $a_i \geq 0$. Then $\langle \lambda\,,\,\alpha\rangle = \sum_{i\in I_P} a_i \langle \lambda\,,\,\alpha_i\rangle \leq -a_j + \sum_{i \in I_P \setminus \{j\}} a_i \langle \lambda\,,\,\alpha_i\rangle \leq -a_j \leq -1$. Therefore $\langle \lambda\,,\,\alpha_i\rangle = 0$ for all $i \in I_P \setminus \{j\}$ and $a_j = 1$. (1) and (2) follow. For (3) we have $\mathrm{Inv}(w) \cap R_P^+ = \mathrm{Inv}(w_2)$. But $\mathrm{Inv}(w_2)$ must consist of the set of roots of R_P^+ in which α_j occurs. This is precisely $\mathrm{Inv}(v_j^P)$. Hence $w_2 = v_j^P$. □

Lemma 10.3. *Suppose* $wt_\lambda \in (W^P)_{\mathrm{af}}$ *and* $w = w_1 w_2 \in W$ *where* $w_1 \in W^P$ *and* $w_2 \in W_P$. *Then* w_2 *has the following form. Let* $J = \{j \in I_P \mid \langle \lambda\,,\,\alpha_j\rangle = -1\}$. *Then* $|J \cap I_m| \leq 1$ *for all* m. *If it is nonempty call this element* j_m*; it is cominuscule in* I_m. *If it is empty write* $j_m = 0_m \in (I_m)_{\mathrm{af}}$. *Then* $w_2 = \prod_{m=1}^k v_{j_m}^{I_m}$.

Proof. This follows from Lemma 10.2. □

Lemma 10.4. *Let* $\alpha \in R_{\mathrm{af}}^+$ *be a real root. Then* $r_\alpha \in (W_P)_{\mathrm{af}}$ *if and only if* $\bar\alpha \in R_P$.

Proof. Follows from (2). □

Lemma 10.5. [22] *For every* $w \in W_{\mathrm{af}}$ *there is a unique factorization* $w = w_1 w_2$ *for* $w_1 \in (W^P)_{\mathrm{af}}$ *and* $w_2 \in (W_P)_{\mathrm{af}}$.

Proof. For existence we may assume that $w\alpha < 0$ for some $\alpha \in R_{\mathrm{af}}^+$ such that $\bar\alpha \in R_P$. Then $wr_\alpha < w$ and by Lemma 10.4 we have $r_\alpha \in (W_P)_{\mathrm{af}}$. By induction $wr_\alpha = x_1 x_2$ with $x_1 \in (W^P)_{\mathrm{af}}$ and $x_2 \in (W_P)_{\mathrm{af}}$. Then $w = x_1(x_2 r_\alpha)$ as desired.

For uniqueness, suppose $w = w_1 w_2 = w_1' w_2'$ with $w_1, w_1' \in (W^P)_{\mathrm{af}}$ and $w_2, w_2' \in (W_P)_{\mathrm{af}}$. Then $w_1 w_2 (w_2')^{-1} = w_1' \in (W^P)_{\mathrm{af}}$. Let $v = w_2(w_2')^{-1} \in (W_P)_{\mathrm{af}}$. If $v \neq 1$ then there is some $\beta \in R_{\mathrm{af}}^+$ such that $\overline{\beta} \in R_P$ and $v\beta < 0$. But $\overline{v\beta} \in R_P$. Since $w_1 \in (W^P)_{\mathrm{af}}$, we have $w_1 v \cdot \beta < 0$, contradicting the assumption that $w_1 v = w_1' \in (W^P)_{\mathrm{af}}$. Uniqueness follows. □

Define $\pi_P : W_{\mathrm{af}} \to (W^P)_{\mathrm{af}}$ by $w \mapsto w_1$ in the notation of Lemma 10.5.

Lemma 10.6. *Let* $\psi_P : Q^\vee \to P_P^\vee$ *be the linear map defined by*

$$\psi_P(\lambda) = \sum_{j \in I_P} \langle \lambda\,,\,\alpha_j\rangle \omega_j^\vee.$$

Let

$$\psi_P(\lambda) + Q_P^\vee \mapsto (-\omega_{j_1}^\vee + Q_1^\vee, \dots, -\omega_{j_k}^\vee + Q_k^\vee)$$

under the isomorphism (10) *and define*

$$\phi_P(\lambda) = -\psi_P(\lambda) - \sum_{m=1}^k \omega_{j_m}^\vee \in Q_P^\vee. \tag{11}$$

Then

$$\pi_P(t_\lambda) = vt_{\lambda + \phi_P(\lambda)}$$

where $v = \prod_{m=1}^k v_{j_m}^{I_m} \in W_P$.

Proof. Since $\phi_P(\lambda) \in Q_P^\vee$, by definition $\pi_P(t_\lambda) = \pi_P(t_{\lambda+\phi_P(\lambda)})$. We have

$$v(\lambda + \phi_P(\lambda)) - (\lambda + \phi_P(\lambda)) = \sum_m (\omega_{j_m}^\vee - v_{j_m}^{I_m} \omega_{j_m}^\vee) \in Q_P^\vee.$$

Therefore

$$\pi_P(t_{\lambda+\phi_P(\lambda)}) = \pi_P(t_{v(\lambda+\phi_P(\lambda))}) = \pi_P(vt_{\lambda+\phi_P(\lambda)}v^{-1}) = \pi_P(vt_{\lambda+\phi_P(\lambda)}).$$

It suffices to show that $vt_{\lambda+\phi_P(\lambda)} \in (W^P)_{\mathrm{af}}$. To this end, let $\alpha + n\delta \in (R_P)^+_{\mathrm{af}}$. We have $\alpha \in R_p$ for some $1 \le p \le k$. Then

$$\begin{aligned} vt_{\lambda+\phi_P(\lambda)}(\alpha + n\delta) &= v\alpha + (n + \sum_{m=1}^k \langle \omega_{j_m}^\vee\,,\,\alpha\rangle)\delta \\ &= v_{j_p}^{I_p}\alpha + (n + \langle \omega_{j_p}^\vee\,,\,\alpha\rangle)\delta. \end{aligned}$$

If $j_p = 0_p$ then $v_{j_p}^{I_p} = 1$, $\omega_{j_p}^\vee = 0$, and $vt_{\lambda+\phi_P(\lambda)}(\alpha + n\delta) = \alpha + n\delta \in R^+_{\mathrm{af}}$. If $j_p \neq 0_p$ then $\langle \omega_{j_p}^\vee\,,\,\alpha\rangle = 1$ and $vt_{\lambda+\phi_P(\lambda)}(\alpha + n\delta) = v_{j_p}^{I_p}\alpha + (n+1)\delta \in R^+_{\mathrm{af}}$ as desired. □

Lemma 10.7. *Suppose* $\lambda \in \tilde{Q}$ *is antidominant. Then* $\phi_P(\lambda)$ *is a non-negative sum of positive coroots* $\{\alpha_i^\vee \mid i \in I_P\}$.

Proof. We may suppose P is irreducible. If $\lambda \in \tilde{Q}$ then $\mu = -\psi_P(\lambda) \in P_P^\vee$ is a dominant coweight. But it is well known (see [10, Section 13]) that $\mu - \omega_i^\vee$ is a sum of positive coroots for some cominuscule node $i \in I_P^s$. □

Example 10.1. We compute some examples of $\pi_P(t_\lambda)$ using Lemma 10.6, working within the subsystem R_P.

(1) In type A_3 let $I_P = \{2,3\} = I_1$ and $\lambda = -\alpha_1^\vee$. R_P is an irreducible subsystem of type A_2. We have $\psi_P(-\alpha_1^\vee) = \omega_2^\vee \in P_P^\vee$ and $\omega_2^\vee = -\omega_3^\vee + (\alpha_2^\vee + \alpha_3^\vee)$. Therefore $j_1 = 3$, $v_3 = r_2r_3$, $\phi_P(-\alpha_1^\vee) = -\alpha_2^\vee - \alpha_3^\vee$, and $\pi_P(t_{-\alpha_1^\vee}) = r_2r_3t_{-\alpha_1^\vee-\alpha_2^\vee-\alpha_3^\vee} = r_2r_3t_{-\theta^\vee}$ where $\theta^\vee$ is the coroot associated to the highest root θ.

Doing this another way, we have $-\alpha_1^\vee = -r_2r_3\theta^\vee$, so that $t_{-\alpha_1^\vee} = r_2r_3t_{-\theta^\vee}r_3r_2$. Removing the right factor $r_3r_2 \in (W_P)_{\mathrm{af}}$ we obtain $r_2r_3t_{-\theta^\vee}$.

(2) In type A_3 let $I_P = \{1,3\}$ and $\lambda = -\alpha_2^\vee$. Then $I_P = I_1 \sqcup I_2$ with $I_1 = \{1\}$ and $I_2 = \{3\}$ with R_1 and R_2 both of type A_1. We have $\psi_P(-\alpha_2^\vee) = \omega_1^\vee + \omega_3^\vee \in P_P^\vee$. We have $\omega_1^\vee = -\omega_1^\vee + \alpha_1^\vee$ and $v_1 = r_1$ in R_1 and $\omega_3^\vee = -\omega_3^\vee + \alpha_3^\vee$ and $v_3 = r_3$ in R_2. Therefore $\phi_P(\lambda) = -\alpha_1^\vee - \alpha_3^\vee$ and $\pi_P(t_{-\alpha_2^\vee}) = r_1r_3t_{-\alpha_1^\vee-\alpha_2^\vee-\alpha_3^\vee} = r_1r_3t_{-\theta^\vee}$.

Another way, we have $t_{-\alpha_2^\vee} = r_1r_3t_{-\theta^\vee}r_3r_1$, and removing the right factor $r_3r_1 \in (W_P)_{\mathrm{af}}$ we have $r_1r_3t_{-\theta^\vee}$ as desired.

(3) In type C_3 with α_3 the long root, let $I_P = \{2,3\} = I_1$ so that R_P is an irreducible subsystem of type C_2. Let $\lambda = -\alpha_1^\vee$. Then $\psi_P(-\alpha_1^\vee) = \omega_2^\vee$. But in R_P we have $\omega_2^\vee = \alpha_2^\vee + \alpha_3^\vee$. In particular $j_1 = 0$ and $\pi_P(t_{-\alpha_1^\vee}) = t_{-\alpha_1^\vee-\alpha_2^\vee-\alpha_3^\vee} = t_{-\theta^\vee}$.

Another way, we have $-\alpha_1^\vee = -\theta^\vee + r_1\theta^\vee$. Therefore we get $t_{-\alpha_1^\vee} = t_{-\theta^\vee}r_1t_{\theta^\vee}r_1 = r_\theta r_0r_1r_0r_\theta r_1 = r_{(12321)010(12321)1} = r_{1232010232}$. Because $r_2r_3r_2 \in (W_P)_{\mathrm{af}}$ we can remove this right factor. $r_{1232010}$ has inversion

$\delta - 2\alpha_2 - \alpha_3 = \alpha_0 + 2\alpha_1 = r_1(\alpha_0)$, so $r_{101} = r_{\delta-2\alpha_2-\alpha_3}$ and $r_{1232010}r_{101} = r_{123210} = t_{-\theta^\vee}$ as desired.

(4) In type B_3 with α_3 the short root, let $I_P = \{2,3\} = I_1$ so that R_P is irreducible of type B_2. Let $\lambda = -\alpha_1^\vee$. We have $\psi_P(-\alpha_1^\vee) = \omega_2^\vee$. We have $\omega_2^\vee = -\omega_2^\vee + 2\alpha_2^\vee + \alpha_3^\vee$. Therefore $j_1 = 2$, $v_1 = r_2r_3r_2$, $\phi_P(\lambda) = -2\alpha_2^\vee - \alpha_3^\vee$, and $\pi_P(t_{-\alpha_1^\vee}) = r_2r_3r_2t_{-\alpha_1^\vee-2\alpha_2^\vee-\alpha_3^\vee} = r_2r_3r_2t_{-\theta^\vee}$.

Another way, $-\alpha_1^\vee = -r_2r_3r_2\theta^\vee$, so $t_{-\alpha_1^\vee} = r_2r_3r_2t_{-\theta^\vee}r_2r_3r_2$. Removing the right factor $r_2r_3r_2 \in (W_P)_{\mathrm{af}}$ we obtain $r_2r_3r_2t_{-\theta^\vee}$ as desired.

Proposition 10.8. [22] *Let $z \in W_{\mathrm{af}}$, $\beta \in R_{\mathrm{af}}^+$, and $\lambda \in Q^\vee$.*

(1) *$\pi_P(W) \subset W^P \subset (W^P)_{\mathrm{af}} \subset (W_{\mathrm{af}})^P$ where $(W_{\mathrm{af}})^P$ is the set of minimum length coset representatives for W_{af}/W_P.*
(2) *$\pi_P(W_{\mathrm{af}}^-) \subset W_{\mathrm{af}}^-$.*
(3) *$\pi_P(z) \le z$.*
(4) *$\pi_P(zt_\lambda) = \pi_P(z)\pi_P(t_\lambda)$.*

Proof. (1) follows from the definitions. (3) follows from the proof of Lemma 10.5.

We first check (4) for $z \in W$. Note that $\pi_P(z) = z_1$ where $z = z_1z_2$ is such that $z_1 \in W^P$ and $z_2 \in W_P$. We have $zt_\lambda = z_1t_{z_2\cdot\lambda}z_2$. Since $z_2 \in W_P$ we have $\lambda - z_2\cdot\lambda \in Q_P^\vee$. It follows that $\pi_P(zt_\lambda) = \pi_P(z_1t_\lambda)$. But $\pi_P(t_\lambda)$ stabilizes $(R_P)_{\mathrm{af}}^+$ by the proof of Lemma 10.6, and $z_1 \in W^P$ has no inversions in $(R_P)_{\mathrm{af}}^+$. Therefore $z_1\pi_P(t_\lambda) \in (W^P)_{\mathrm{af}}$, which finishes the proof of (4) for $z \in W$. Using this we may reduce the proof of (4) for $z \in W_{\mathrm{af}}$, to the case that $z = t_{\lambda'}$ for some $\lambda' \in Q^\vee$. Since $\pi_P(t_\lambda)$ stabilizes $(R_P)_{\mathrm{af}}^+$ it follows that $\pi_P(t_{\lambda'})\pi_P(t_\lambda) \in (W^P)_{\mathrm{af}}$. Therefore it is enough to show that $\pi_P(t_{\lambda'+\lambda})$ and $\pi_P(t_{\lambda'})\pi_P(t_\lambda)$ differ by a right multiple of t_μ for some $\mu \in Q_P^\vee$. By Lemma 10.6 there exist $v', v'' \in W_P$ such that $\pi_P(t_{\lambda'}) = v't_{\lambda'+\phi_P(\lambda')}$ and $\pi_P(t_{\lambda'+\lambda}) = v''t_{\lambda'+\lambda+\phi_P(\lambda'+\lambda)}$. We have

$$\begin{aligned}\pi_P(t_{\lambda'})\pi_P(t_\lambda) &= v't_{\lambda'+\phi_P(\lambda')}vt_{\lambda+\phi_P(\lambda)} \\ &= v'vt_{v(\lambda'+\phi_P(\lambda'))+\lambda+\phi_P(\lambda)}.\end{aligned}$$

But the map $Q^\vee \to W_P$ given by $\lambda \mapsto v$, where $v \in W_P$ is such that $\pi_P(t_\lambda) = vt_{\lambda+\phi_P(\lambda)}$, is a group homomorphism, that is, $v'' = v'v$. Moreover $\lambda' + \phi_P(\lambda')$ and its image under $v \in W_P$, differ by an element of $Q_P^\vee$. Therefore (4) follows.

For (2), let $x = wt_\lambda \in W_{\mathrm{af}}^-$ for $\lambda \in \tilde{Q}$. Then $\pi_P(t_\lambda) = vt_{\lambda+\phi_P(\lambda)}$ and $\pi_P(x) = \pi_P(w)\pi_P(t_\lambda)$. To show that $\pi_P(x) \in W_{\mathrm{af}}^-$ we check that $\pi_P(x)\cdot\alpha_i > 0$ for each $i \in I$. We will repeatedly use the following criterion: $ut_\mu\cdot\alpha_i > 0$ if and only if either $\langle\mu\,,\,\alpha_i\rangle < 0$ or $\langle\mu\,,\,\alpha_i\rangle = 0$ and $\alpha_i \notin \mathrm{Inv}(u)$. In particular we need to establish one of these conditions for $u = \pi_P(w)v$ and $\mu = \lambda + \phi_P(\lambda)$.

Suppose first that $i \in I_P$. Then by Lemma 10.1, $\langle\lambda+\phi_P(\lambda)\,,\,\alpha_i\rangle = -1$ or 0 and in the case of 0 we have $\alpha_i \notin \mathrm{Inv}(v)$ and thus $\alpha_i \notin \mathrm{Inv}(\pi_P(w)v)$. In either case we are done.

Otherwise suppose that $i \notin I_P$ and that the Dynkin node i is not connected to any node in I_P. Then $\langle\lambda+\phi_P(\lambda)\,,\,\alpha_i\rangle = \langle\lambda\,,\,\alpha_i\rangle$ and $\alpha_i \in \mathrm{Inv}(w) \Leftrightarrow \alpha_i \in \mathrm{Inv}(\pi_P(w))$. Since $x\cdot\alpha_i > 0$ we conclude that $\pi_P(x)\cdot\alpha_i > 0$.

Finally suppose that $i \notin I_P$ and that the set J of nodes in I_P connected to i, is nonempty. By Lemma 10.7, $\langle\lambda+\phi_P(\lambda)\,,\,\alpha_i\rangle \le \langle\lambda\,,\,\alpha_i\rangle$. We are immediately done if $\langle\lambda\,,\,\alpha_i\rangle < 0$ or $\langle\lambda\,,\,\alpha_i\rangle = 0$ and $\langle\phi_P(\lambda)\,,\,\alpha_i\rangle < 0$. Suppose otherwise, so that $\phi_P(\lambda)$ does not involve any roots α_j where $j \in J$.

We know by Lemma 10.1 that $\langle \lambda + \phi_P(\lambda)\,,\, \alpha_j \rangle = -1$ or 0. Suppose for some $j \in J$ that $\langle \lambda + \phi_P(\lambda)\,,\, \alpha_j \rangle = 0$. Then since $\phi_P(\lambda)$ does not involve α_j, we have $\langle \lambda\,,\, \alpha_j \rangle = 0 = \langle \phi_P(\lambda)\,,\, \alpha_j \rangle$. Let P' be such that $I_{P'} = I_P \setminus \{j\}$. We may suppose inductively that $\pi_{P'}(x) \in W_{\mathrm{af}}^-$. We claim that $\pi_{P'}(x) = \pi_P(x)$. Since $(W_{P'})_{\mathrm{af}} \subset (W_P)_{\mathrm{af}}$ it suffices to show that $\pi_{P'}(x) \in (W^P)_{\mathrm{af}}$. We first note that by our assumptions $\phi_P(\lambda) = \phi_{P'}(\lambda)$ (using the fact that a cominuscule node in a component of I_P is still cominuscule in $I_{P'}$). Let $\pi_{P'}(x) = ut_{\lambda+\phi_P(\lambda)}$. Since $\langle \lambda + \phi_{P'}(\lambda)\,,\, \alpha_j \rangle = 0$ and $\pi_{P'}(x) \in W_{\mathrm{af}}^-$ we have $u \cdot \alpha_j > 0$. We can thus deduce using Lemmata 10.1 and 10.2 that $\pi_{P'}(x) \in (W^P)_{\mathrm{af}}$.

Thus we may assume for our chosen $i \in I_P$ (with $\langle \lambda\,,\, \alpha_i \rangle = 0$) that all $j \in J$ satisfy $\langle \lambda + \phi_P(\lambda)\,,\, \alpha_j \rangle = -1$. Note that these j all lie in different connected components of I_P (thus $|J| \in \{1, 2, 3\}$). We need to show that $\pi_P(w)v \cdot \alpha_i > 0$. We may assume that I_P is exactly the union of the connected components $I_{P_j} \subset I_P$ containing each $j \in J$, so that $v = \prod_{j\in J} v_j$ where $v_j \in W_{P_j}$ are the elements described in Lemma 10.2. For each parabolic subgroup $W_Q \subset W$, write $w_Q \in W_Q$ for its longest element. Then by definition $v_j = w_{P_j} w_{P'_j}$ where $P'_j = P_j \setminus \{j\}$. Also factorize $\pi_P(w)$ as $u'u$ where u lies in the parabolic subgroup $W' \subset W$ corresponding to the nodes $\{i\} \cup I_P$ and u' is minimal length in W/W'. It suffices to show that $uv \cdot \alpha_i > 0$. We calculate that

$$uv \cdot \alpha_i = u \prod_j w_{P_j} w_{P'_j} \cdot \alpha_i = uw_P \cdot \alpha_i.$$

But $u \in (W')^P$ so that uw_P is a length-additive factorization as $u \in (W')^P$ and $w_P \in (W')_P = W_P$. We know $w_P \cdot \alpha_k < 0$ for $k \in I_P$. If $uw_P \cdot \alpha_i < 0$ as well then we must have $uw_P = w'_0$, the longest element in W'. But w factorizes uniquely (and length-additively) as $u'(uu'')$ where $u'' \in W_P$. If $u = w'_0 w_P$ then $uu'' \cdot \alpha_i < 0$ which in turn means $w \cdot \alpha_i < 0$, contradicting the assumption that $x = wt_\lambda \in W_{\mathrm{af}}^-$. □

10.4. **Ideals of $H_T(\mathrm{Gr}_G)$.**

Proposition 10.9 ([22]). *For $\alpha \in R_{\mathrm{af}}^+$, the S-submodule*

$$K(\alpha) = \bigoplus_{\substack{x \in W_{\mathrm{af}}^- \\ x\cdot\alpha<0}} S\,\xi_x$$

of $H_T(\mathrm{Gr}_G)$, is an ideal of $H_T(\mathrm{Gr}_G)$.

Proof. By (6) it suffices to show that $K(\alpha)$ has a left $\mathbb{A}_{\mathrm{af}}$-action. By (5) it suffices to show that if $x \in W_{\mathrm{af}}^-$, $r_i x > x$, and $x\alpha < 0$, then $r_i x\alpha < 0$. Suppose not, that is, $r_i x \alpha > 0$. Then $x\alpha = -\alpha_i$ and $0 > -\alpha = x^{-1}\alpha_i$. But $x^{-1} < x^{-1} r_i$ so that $x^{-1}\alpha_i > 0$, a contradiction. □

Thus

$$J_P = \sum_{\alpha \in (R_P)_{\mathrm{af}}^+} K(\alpha) = \sum_{x \in W_{\mathrm{af}}^- \setminus (W^P)_{\mathrm{af}}} S\xi_x$$

is an ideal of $H_T(\mathrm{Gr}_G)$.

10.5. **Parabolic quantum parameters.**

Lemma 10.10. *Let $\lambda \in \tilde{Q}$. Then $A_i \cdot \xi_{\pi_P(t_\lambda)} = 0 \mod J_P$ for each $i \in I$.*

Proof. By (5) $A_i \cdot \xi_{\pi_P(t_\lambda)} = 0$ unless $\ell(r_i\, \pi_P(t_\lambda)) = \ell(\pi_P(t_\lambda)) + 1$ and $r_i\, \pi_P(t_\lambda) \in W_{\mathrm{af}}^-$. By Lemma 10.6, $\pi_P(t_\lambda) = vt_\nu$ for some $v \in W_P$ and $\nu \in \tilde{Q}$.

Suppose $i \notin I_P$. Then $\ell(r_i v) = \ell(v) + 1$ and by Lemma 3.3 $\ell(r_i v t_\nu) = \ell(vt_\nu) - 1$, so $A_i \cdot \xi_{\pi_P(t_\lambda)} = 0$.

Suppose $i \in I_P$. Then $r_i v \in W_P$. By Lemma 10.3 we have $r_i v t_\nu \notin (W^P)_{\mathrm{af}}$ and $\xi_{\pi_P(t_\lambda)} = 0 \mod J_P$. □

Note that we exclude $i = 0$ in Lemma 10.10. The following result generalizes Proposition 9.1.

Proposition 10.11. *Let $x \in W_{\mathrm{af}}^- \cap (W^P)_{\mathrm{af}}$ and $\lambda \in \tilde{Q}$. Then $x\pi_P(t_\lambda) \in W_{\mathrm{af}}^- \cap (W^P)_{\mathrm{af}}$ and we have*

$$\xi_x\, \xi_{\pi_P(t_\lambda)} = \xi_{x\pi_P(t_\lambda)} \mod J_P.$$

Proof. By Lemma 10.10, $J \cdot \xi_{\pi_P(t_\lambda)} = 0 \mod J_P$, where $J = \sum_{w \in W\setminus\{\mathrm{id}\}} \mathbb{A}_{\mathrm{af}} A_w$ as in Theorem 6.2. By Theorem 6.2 we thus have

$$\xi_x\, \xi_{\pi_P(t_\lambda)} = A_x \cdot \xi_{\pi_P(t_\lambda)} \mod J_P.$$

It suffices thus to show that the product $x\, \pi_P(t_\lambda)$ is length-additive. Since $x \in W_{\mathrm{af}}^- \cap (W^P)_{\mathrm{af}}$ using Proposition 10.8 we may write $x = w\pi_P(t_\nu)$ for $w \in W^P$ and $\nu \in \tilde{Q}$. We have $\ell(w\pi_P(t_\mu)) = -\ell(w) + \ell(\pi_P(t_\mu))$ for every $\mu \in \tilde{Q}$ such that $wt_\mu \in W_{\mathrm{af}}^-$, so it suffices to show that $\ell(\pi_P(t_{\nu+\lambda})) = \ell(\pi_P(t_\lambda)) + \ell(\pi_P(t_\nu))$ for $\nu, \lambda \in \tilde{Q}$. By Lemma 10.6 we may assume that ν, λ are chosen so that $\pi_P(t_\nu) = v_\nu\, t_\nu$ and $\pi_P(t_\lambda) = v_\lambda\, t_\lambda$. By Lemma 10.3, $\ell(v_\lambda) = -\langle \lambda, 2\rho_P\rangle$ where $2\rho_P = \sum_{\alpha \in R_P^+} \alpha$ and similarly for v_ν. Thus by Lemma 3.3, $\ell(v_\lambda\, t_\lambda) = -\langle \lambda, 2(\rho - \rho_P)\rangle$ and similarly for ν and $\nu + \lambda$.

The last statement follows immediately from Proposition 10.8 since $\pi_P(xt_\lambda) = x\pi_P(t_\lambda)$. □

10.6. **Quantum parabolic Chevalley formula.** The equivariant quantum cohomology $QH^T(G/P)$ is the free $S[q_i \mid i \in I \setminus I_P]$-module spanned by the equivariant quantum Schubert classes $\{\sigma_P^w \mid w \in W^P\}$. For $\lambda = \sum_i a_i \alpha_i^\vee \in Q^\vee/Q_P^\vee$ with $a_i \in \mathbb{Z}$ we let $q_\lambda = \prod_{i \in I \setminus I_P} q_i^{a_i}$. The quantum multiplication of $QH^T(G/P)$ is denoted again with $*$.

Recall that for $w \in W$, if we write $w = w_1 w_2$ with $w_1 \in W^P$ and $w_2 \in W_P$ then $w_1 = \pi_P(w)$. Recall that $2\rho_P = \sum_{\alpha \in R_P^+} \alpha$. Let $\eta_P : Q^\vee \to Q^\vee/Q_P^\vee$ be the natural projection.

Theorem 10.12 (Quantum equivariant parabolic Chevalley formula [21])**.** *Let $i \in I \setminus I_P$ and $w \in W^P$. Then we have in $QH^T(G/P)$*

$$\begin{aligned}\sigma_P^{r_i} * \sigma_P^w = (\omega_i - w \cdot \omega_i)\sigma_P^w &+ \sum_\alpha \langle \alpha^\vee, \omega_i\rangle\, \sigma_P^{wr_\alpha} \\ &+ \sum_\alpha \langle \alpha^\vee, \omega_i\rangle\, q_{\eta_P(\alpha^\vee)}\, \sigma^{\pi_P(wr_\alpha)}\end{aligned}$$

where the first summation is over $\alpha \in R^+ \setminus R_P^+$ such that $wr_\alpha \gtrdot w$ and $wr_\alpha \in W^P$, and the second summation is over $\alpha \in R^+ \setminus R_P^+$ such that $\ell(\pi_P(wr_\alpha)) = \ell(w) + 1 - \langle \alpha^\vee, 2(\rho - \rho_P)\rangle$.

Mihalcea [21] showed that the quantum equivariant parabolic Chevalley formula completely determines the multiplication in $QH^T(G/P)$.

We will use a special case of the Peterson-Woodward comparison formula to clarify the second summation in Theorem 10.12. For $u, v, w \in W^P$ and $\lambda \in Q^\vee/Q_P^\vee$ let $d_{u,v}^{w,\lambda,P}$ denote the coefficient of $q_\lambda \sigma_P^w$ in $\sigma_P^u * \sigma_P^v$, calculated in $QH^*(G/P)$. We use $d_{u,v}^{w,\lambda,P}$ instead of $c_{u,v}^{w,\lambda,P}$ since Woodward's result is stated only for the non-equivariant coefficients.

Theorem 10.13 ([25, Lemma 1, Thm. 2])**.**

(1) *For every $\lambda_P \in Q^\vee/Q_P^\vee$ there exists a unique $\lambda_B \in Q^\vee$ such that $\eta_P(\lambda_B) = \lambda_P$ and $\langle \lambda_B\,,\,\alpha\rangle \in \{0,-1\}$ for all $\alpha \in R_P^+$. Moreover if $\langle \lambda_P\,,\,\alpha_i\rangle \le 0$ for $i \in I \setminus I_P$ then $\langle \lambda_B\,,\,\alpha_i\rangle \le 0$ for all $i \in I$.*

(2) *For every $x, y, z \in W^P$ we have*

$$d_{x,y}^{z,\lambda_P,P} = d_{x,y}^{z\,w_P\,w_{P'},\lambda_B}$$

where w_P is the longest element in W_P and $P' \subset P$ is the standard parabolic subgroup of P such that $I_{P'} = \{i \in I_P \mid \langle \lambda_B\,,\,\alpha_i\rangle = 0\}$.

Remark 10.2. In [25], Theorem 10.13 is stated instead in terms of the coefficients $\langle x, y, w_0\, z\, w_P\rangle_{\lambda_P} = d_{xy}^{z,\lambda_B,P}$. Since $w_B = \mathrm{id}$, our formulation is recovered.

Remark 10.3. In Theorem 10.13, λ_B and P' may be computed explicitly. Given $\lambda_P \in Q^\vee/Q_P^\vee$, let $\lambda \in Q^\vee$ be defined by $\lambda = \sum_{i \in I\setminus I_P} \langle \lambda_P\,,\,\omega_i\rangle \alpha_i^\vee$; it clearly satisfies $\eta_P(\lambda) = \lambda_P$. Let $\pi_P(t_\lambda) = v t_{\lambda+\phi_P(\lambda)}$ be as in Lemma 10.6. Then $\lambda_B = \lambda + \phi_P(\lambda)$, $I_{P'} = I_P \setminus \{j_m \mid 1 \le m \le k \text{ and } j_m \neq 0_m\}$, and $v = w_P w_{P'}$.

Lemma 10.14. *The second summation in Theorem 10.12 is over $\alpha \in R^+ \setminus R_P^+$ such that*

(1) *$\ell(\pi_P(w r_\alpha)) = \ell(w) + 1 - \langle \alpha^\vee, 2(\rho - \rho_P)\rangle$, and*
(2) *$\ell(w r_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho\rangle + 1$.*

Proof. Using the notation of Theorems 10.12 and 10.13 set $x = r_i$, $y = w$, $z = \pi_P(w r_\alpha)$, and $\lambda_P = \eta_P(\alpha^\vee)$. Then the coefficient of $q_{\eta_P(\alpha^\vee)} \sigma_P^{\pi_P(w r_\alpha)}$ in $\sigma_P^{r_i} * \sigma_P^w$ is 0 unless the coefficient of $q_{\lambda_B} \sigma^{\pi_P(w r_\alpha)\, w_P\, w_{P'}}$ in $\sigma^{r_i} * \sigma^w$ is non-zero.

By the Claim within Lemma 4.1 of [8], we know that $\pi_P(r_\alpha) \neq \pi_P(r_\beta)$ for any $\alpha \neq \beta$ both in $R^+ \setminus R_P^+$. Since $\pi_P(w r_\alpha)\, w_P\, w_{P'} W_P = w r_\alpha W_P$ we conclude that the coefficient of $\sigma_P^{\pi_P(w r_\alpha)}$ in $\sigma_P^{r_i} * \sigma_P^w$ is non-zero only if $\sigma^{w r_\alpha}$ occurs in $\sigma^{r_i} * \sigma^w$. By Theorem 2.1 and the last statement of Theorem 10.13, the latter holds only if $\ell(w r_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho\rangle + 1$. ∎

Remark 10.4. Presumably Lemma 10.14 can be deduced from Theorem 10.12 purely Coxeter-theoretically; that is, without the additional input provided by Theorem 10.13.

10.7. **Parabolic Peterson Theorem.**

Lemma 10.15. *The map $\pi_P(t_\nu) \mapsto \eta_P(\nu)$ is a bijection onto $Q^\vee/Q_P^\vee$.*

Proof. By definition, $\pi_P(t_\nu) = \pi_P(t_{\nu+\mu})$ if $\mu \in Q_P^\vee$. Thus the map is well defined and clearly a surjection. By Proposition 10.8, it thus suffices to show that if $\eta_P(\nu) = 0$ then $\pi_P(t_\nu) = \mathrm{id}$. But $\eta_P(\nu) = 0$ means that $\nu \in Q_P^\vee$ so $t_\nu \in (W_P)_{\mathrm{af}}$ and $\pi_P(t_\nu) = \mathrm{id}$. ∎

Theorem 10.16. *There is an S-algebra isomorphism*

$$\Psi_P : (H_T(\mathrm{Gr}_G)/J_P)[\xi^{-1}_{\pi_P(t_\lambda)} \mid \lambda \in \tilde{Q}] \longrightarrow QH^T(G/P)[q_i^{-1} \mid i \in I \setminus I_P]$$
$$\xi_{v\pi_P(t_\lambda)}\, \xi^{-1}_{\pi_P(t_\nu)} \longmapsto q_{\eta_P(\lambda-\nu)}\, \sigma_P^v$$

for $v \in W^P$ and $\lambda, \nu \in \tilde{Q}$.

Proof. Using Lemma 10.15, the map Ψ_P is easily seen to be an isomorphism of S-modules. Since the quantum parabolic Chevalley formula determines the ring structure of $QH^T(G/P)$, it suffices to prove that the Ψ_P-preimage of this relation holds in $H_T(\mathrm{Gr}_G)/J_P$. By Proposition 10.11, it suffices to check the product $\xi_{v\pi_P(t_\lambda)}\, \xi_{r_i\pi_P(t_\nu)}$ for a choice of $\nu, \lambda \in \tilde{Q}$ for each $i \in I \setminus I_P$ and $v \in W^P$. Taking a large power of $\pi_P(t_\lambda)$ and using Proposition 10.8, we may choose ν, λ such that $\pi_P(t_\nu) = t_\nu$ and $\pi_P(t_\lambda) = t_\lambda$. By Theorem 9.2, this reduces to checking that the preimage (in the Borel case) of the quantum equivariant Chevalley formula in $H_T(\mathrm{Gr}_G)$, gives rise to that of the quantum equivariant parabolic Chevalley formula after quotienting out by the ideal $J_P \subset H_T(\mathrm{Gr}_G)$.

The equivariant term and the non-quantum terms trivially agree, so we check the quantum terms. For $w \in W^P$ define

$$A_w = \{\alpha \in R^+ \setminus R_P^+ \mid \ell(wr_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho\rangle + 1 \text{ and }$$
$$\ell(\pi_P(wr_\alpha)) = \ell(w) + 1 - \langle \alpha^\vee, 2\rho - 2\rho_P\rangle\}$$

and

$$B_w = \{\alpha \in R^+ \setminus R_P^+ \mid \ell(wr_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho\rangle + 1 \text{ and }$$
$$\pi_P(wr_\alpha t_\alpha) = wr_\alpha t_\alpha\}.$$

Note that A_w indexes quantum terms in the parabolic quantum Chevalley formula by Lemma 10.14 and B_w indexes quantum terms in the preimage of the quantum Borel Chevalley formula in $H_T(\mathrm{Gr}_G)$ which do not vanish modulo J_P.

By Lemma 3.5, the condition $\ell(wr_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho\rangle + 1$ implies that $\ell(wr_\alpha) = \ell(w) - \ell(r_\alpha)$ and $\ell(r_\alpha) = \langle \alpha^\vee, 2\rho\rangle - 1$. The equation $\ell(wr_\alpha) = \ell(w) - \ell(r_\alpha)$ in turn implies that $r_\alpha \in W^P$, since $w \in W^P$. Thus $\langle \alpha^\vee, \beta\rangle \leq 0$ for $\beta \in R_P^+$. Let $x = wr_\alpha = yx'$ with $y = \pi_P(wr_\alpha) \in W^P$ and $x' \in W_P$.

Let $\alpha \in A_w$, that is, $\ell(y) = \ell(w) + 1 - \langle \alpha^\vee, 2\rho - 2\rho_P\rangle$. Thus $\ell(x) - \ell(y) = -\langle \alpha^\vee, 2\rho_P\rangle = -\sum_{\beta \in R_P^+} \langle \alpha^\vee, \beta\rangle$. Let us estimate $\ell(x') = |\mathrm{Inv}(x')|$. Since $xr_\alpha = w \in W^P$ we must have $x'\beta > 0$ for $\beta \in R_P^+$ satisfying $r_\alpha\beta = \beta$. Hence $\ell(x) - \ell(y) = \ell(x') = |\mathrm{Inv}(x')| \leq |\{\beta \in R_P^+ \mid \langle \alpha^\vee, \beta\rangle < 0\}| \leq -\langle \alpha^\vee, 2\rho_P\rangle$. Thus we must have $-1 \leq \langle \alpha^\vee, \beta\rangle \leq 0$ for all $\beta \in R_P^+$ and $\mathrm{Inv}(x') = \{\beta \in R_P^+ \mid \langle \alpha^\vee, \beta\rangle = -1\}$. Using Lemma 10.3, we conclude that $x't_{\alpha^\vee} = \pi_P(t_\alpha^\vee) \in (W^P)_{\mathrm{af}}$. This in turn gives $wr_\alpha t_{\alpha^\vee} = xt_{\alpha^\vee} = y(x't_{\alpha^\vee}) = \pi_P(wr_\alpha)\pi_P(t_{\alpha^\vee})$, showing that $\alpha \in B_w$.

For the reverse inclusion $B_w \subset A_w$, one deduces from $\pi_P(wr_\alpha t_\alpha) = wr_\alpha t_\alpha = \pi_P(wr_\alpha)\pi_P(t_\alpha)$ that $x't_\alpha = \pi_P(t_\alpha)$ satisfies the conditions of Lemma 10.3. In particular $\langle \alpha^\vee, 2\rho_P\rangle = -\ell(x')$. This shows that $B_w \subset A_w$.

Finally we note that a term $q_{\alpha^\vee}\sigma^{wr_\alpha}$ (for $w \in B_w$) in the quantum Borel Chevalley formula gives rise to the class $\xi_{wr_\alpha}\xi^{-1}_{t_{-\alpha^\vee}} \in H_T^t(\mathrm{Gr}_G)$ (where $\xi_{wr_\alpha} := \xi_{wr_\alpha t_\lambda}\xi^{-1}_{t_\lambda}$ for appropriate t_λ) which in turn gives rise to the class $q_{\eta_P(\alpha^\vee)}\sigma_P^{\pi_P(wr_\alpha)}$ in $QH^T(G/P)$. $\square$

For $w,v,u \in W^P$ and $\lambda \in Q^\vee/Q_P^\vee$ let $c_{u,v}^{w,\lambda,P}$ denote the coefficient of $q_\lambda \sigma_P^w$ in $\sigma_P^u * \sigma_P^v$, calculated in $QH^T(G/P)$.

Corollary 10.17. *Let $w,v,u \in W^P$ and $\lambda \in Q^\vee/Q_P^\vee$. Pick $\eta,\kappa,\mu \in \tilde{Q}$ so that $x = w\pi_P(t_\eta), y = v\pi_P(t_\kappa), z = u\pi_P(t_\mu) \in W_{\mathrm{af}}^- \cap (W^P)_{\mathrm{af}}$, where $\lambda = \eta_P(\mu - \eta + \kappa)$. Then the equivariant three point Gromov-Witten invariant $c_{u,v}^{w,\lambda,P}$ is equal to the coefficient of ξ_z in the product $\xi_x\,\xi_y \in H_T(\mathrm{Gr}_G)$.*

Note that in Corollary 10.17, the element z is completely determined by x, y and λ.

Remark 10.5. It would be interesting to compare Corollary 10.17 with the work of Buch, Kresch and Tamvakis [4] who exhibit the Gromov-Witten invariants of (classical, orthogonal and Lagrangian) Grassmannians as classical Schubert structure constants.

11. Application to quantum cohomology

For this section we will work in non-equivariant quantum cohomology $QH^*(G/P)$ and homology $H_*(\mathrm{Gr}_G)$.

11.1. **Highest root.** We apply known formulae in $H_*(\mathrm{Gr}_G)$ to obtain new formulae in $QH^*(G/P)$. Let $K = \sum_{i \in I_{\mathrm{af}}} a_i^\vee \alpha_i^\vee$ be the canonical central element for the affine Lie algebra associated to the Lie algebra of G. It satisfies $a_0^\vee = 1$ and $\theta^\vee = \sum_{i\in I} a_i^\vee \alpha_i^\vee$ where $\theta^\vee$ is the coroot associated with the highest root θ. Let j_0 denote the composition of $j : H_*^T(\mathrm{Gr}_G) \to Z_{\mathrm{af}}(S)$ with the evaluation ϕ at 0: $\phi(\sum_w a_w A_w) = \sum_w \phi_0(a_w) A_w$, where $\phi_0 : S \to \mathbb{Z}$ evaluates a polynomial at 0.

Proposition 11.1 ([16])**.** *We have*

$$j_0(\xi_{r_0}) = \sum_{i \in I_{\mathrm{af}}} a_i^\vee A_i.$$

Thus in $H_*(\mathrm{Gr}_G)$, for $x \in W_{\mathrm{af}}^-$ we have

$$\xi_{r_0}\,\xi_x = \sum_{\substack{i \in I_{\mathrm{af}} \\ r_i x > x \\ r_i x \in W_{\mathrm{af}}^-}} a_i^\vee \xi_{r_i x}.$$

Suppose $P \neq G$. By Remark 10.1, $r_0 = r_\theta t_{-\theta^\vee} \in (W^P)_{\mathrm{af}}$.

Proposition 11.2. *Let $w \in W^P$. We have*

$$\sigma_P^{\pi_P(r_\theta)} * \sigma_P^w = q_{\eta_P(\theta^\vee - w^{-1}\theta^\vee)} \sigma_P^{\pi_P(r_\theta w)} + q_{\eta_P(\theta^\vee)} \sum_{\substack{i\in I \\ r_i w < w}} a_i^\vee\, \sigma_P^{r_i w}$$

where the first term is present if and only if $w \cdot \alpha = \theta$ for some $\alpha \in R^+ \setminus R_P^+$.

Proof. Let $x = wt_\lambda \in W_{\mathrm{af}}^- \cap (W^P)_{\mathrm{af}}$ where we assume as in the proof of Theorem 10.16 that $\pi_P(t_\lambda) = t_\lambda$. By Lemma 10.2, we have $\langle \lambda, \alpha_i \rangle = 0$ for $i \in I_P$. Using Lemma 10.6, we may assume in addition that $\langle \lambda, \alpha_i\rangle \neq 0$ for $i \in I \setminus I_P$. Thus by Lemma 3.3, we have $\ell(r_i x) = \ell(x)+1$ and $r_i x \in W_{\mathrm{af}}^-$ if and only if $\ell(r_i w) = \ell(w) - 1$ (which automatically implies that $r_i w \in W^P$).

Now let us consider $r_0 x = r_0 w t_\lambda$. By our assumptions, $t_\lambda \cdot \alpha = \alpha$ for $\alpha \in R_P^+$, and since the only inversion of r_0 is $\alpha_0 = \delta - \theta$, we deduce that $r_0 x \in (W^P)_{\mathrm{af}}$

if and only if $w\alpha \neq \theta$ for $\alpha \in R_P^+$. If $r_0 x \in (W^P)_{\mathrm{af}}$ then $r_0 x = r_\theta t_{-\theta^\vee} w t_\lambda = (r_\theta w) t_{-w^{-1}\theta^\vee + \lambda} = \pi_P(r_\theta w)\pi_P(t_{-w^{-1}\theta^\vee})\pi_P(t_\lambda)$ by Proposition 10.8.

Also note that in the above situation,

$$\begin{aligned}\ell(r_0 x) = \ell(x) + 1 \Leftrightarrow r_0 x > x & \\ \Leftrightarrow x \cdot (n\delta - \alpha) = \delta - \theta & \qquad \text{for some } n\delta - \alpha \in R_{\mathrm{af}}^+ \\ \Leftrightarrow w \cdot \alpha = \theta & \qquad \text{for some } \alpha \in R^+ \setminus R_P^+.\end{aligned}$$

Finally, we observe that in the above situation we automatically have $r_0 x \in W_{\mathrm{af}}^-$ since $x \in W_{\mathrm{af}}^-$.

Using Proposition 11.1, Theorem 10.16 and these observations we obtain in $QH^*(G/P)$

$$\begin{aligned} q_{\eta_P(-\theta^\vee)} \sigma_P^{\pi_P(r_\theta)} * q_{\eta_P(\lambda)} \sigma_P^w = \sum_{\substack{i \in I \\ r_i w < w}} a_i^\vee q_{\eta_P(\lambda)} \sigma_P^{r_i w} \\ + a_0^\vee q_{\eta_P(\lambda - w^{-1}\theta^\vee)} \sigma_P^{\pi_P(r_\theta w)} \end{aligned}$$

where the last term is present if and only if $w \cdot \alpha = \theta$ for some $\alpha \in R^+ \setminus R_P^+$. Dividing both sides by $q_{\eta_P(\lambda - \theta^\vee)}$ and using $a_0^\vee = 1$, we obtain the required statement. $\square$

In the case that P is a maximal parabolic corresponding to a cominuscule node (as in the following section), Proposition 11.2 looks similar to a formula shown to us by Nicolas Perrin (see [6]).

11.2. **Cominuscule case.** In this section we assume that P is a maximal parabolic such that $I \setminus I_P = \{j\}$ where j is a cominuscule Dynkin node.

The map $W \to W$ given by $w \mapsto w^* = w_0 w w_0$, is an involutive isomorphism that sends simple reflections to simple reflections: $r_i \mapsto (r_i)^* = r_{i^*}$ for some $i^* \in I$. The map $i \mapsto i^*$ is an automorphism of the finite Dynkin diagram. There is an associated automorphism of Q given by $\alpha \mapsto \alpha^* := -w_0\alpha$ which satisfies $(\alpha_i)^* = \alpha_{i^*}$ for $i \in I$. For $w \in W$ and $\alpha \in Q$ we have $(w\alpha)^* = w^*\alpha^*$. There is a similar involution on $P^\vee$ that stabilizes $Q^\vee$, thereby defining an involutive automorphism of $\Sigma = P^\vee/Q^\vee$. Since $-w_0\omega_i^\vee = \omega_{i^*}^\vee$ and $\omega_i^\vee \equiv w_0\omega_i^\vee \mod Q^\vee$, the induced automorphism of $P^\vee/Q^\vee$ is given by negation: $\omega_i^\vee + Q^\vee \mapsto -\omega_i^\vee + Q^\vee$.

The finite Dynkin automorphism $I \to I$ given by $i \mapsto i^*$, may be extended to an automorphism of the affine Dynkin diagram by letting $0^* = 0$. This induces an automorphism of W_{af} again denoted $w \mapsto w^*$.

Proposition 11.3. *Define $\vartheta : W^P \to W_{\mathrm{af}}$ by $\vartheta(y) = \tau_j(y)^*$. Then for every $y \in W^P$, $\vartheta(y) \in (W^P)_{\mathrm{af}} \cap W_{\mathrm{af}}^-$, and $\{\xi_{\vartheta(y)} \mid y \in W^P\}$ is a $S[\xi^\pm_{\pi_P(t_\lambda)} \mid \lambda \in \tilde{Q}]$-basis of $(H_T(\mathrm{Gr}_G)/J_P)[\xi^{-1}_{\pi_P(t_\lambda)} \mid \lambda \in \tilde{Q}]$. Moreover if $\vartheta(y) = wt_\lambda$ then $\pi_P(w) = \pi_P(w_0^P y)$.*

Proof. Note that $i \mapsto \tau_j(i)^* = \tau_{j^*}(i^*)$ is an involutive affine Dynkin automorphism that stabilizes $I_{\mathrm{af}} \setminus \{0, j\}$ and exchanges 0 and j. It follows that $\alpha \mapsto \tau_j(\alpha)^* = \tau_{j^*}(\alpha^*)$ stabilizes R_P^+. This map also permutes the affine simple roots and hence stabilizes R_{af}^+.

Let $y \in W^P$. Then $y \cdot \alpha_i > 0$ for all $i \in I_{\mathrm{af}} \setminus \{0, j\}$. Consequently $\vartheta(y) \cdot \alpha_i > 0$ for all $i \in I_{\mathrm{af}} \setminus \{0, j\}$. Since $y \in W$, $\vartheta(y)$ is in the subgroup of W_{af} generated by r_i for $i \in I_{\mathrm{af}} \setminus \{j\}$, so that $\vartheta(y) \cdot \alpha_j > 0$. Therefore $\vartheta(y) \in W_{\mathrm{af}}^-$.

For all $\alpha \in R_P^+$ we have

$$\vartheta(y) \cdot \alpha = (\tau_j(y)\alpha^*)^* = (\tau_j(y\tau_{j^*}(\alpha^*)))^*.$$

We have $\tau_{j^*}(\alpha^*) \in R_P^+$, so that $\beta = y \cdot \tau_{j^*}(\alpha^*) \in R^+$. Since $j \in I$ is cominuscule, α_j has multiplicity at most one in β. Therefore $\tau_j(\beta)^* \in R_{\mathrm{af}}^+$, in which α_0 occurs with multiplicity at most one. It follows that $\tau_j(\beta)^*$ has the form γ or $\delta - \gamma$ for some $\gamma \in R^+$. Therefore $\vartheta(y) \cdot \alpha \in R_{\mathrm{af}}^+$ and $\vartheta(y) \cdot (\delta - \alpha) \in R_{\mathrm{af}}^+$, proving that $\vartheta(y) \in (W^P)_{\mathrm{af}}$.

We have $w_0 r_0 w_0 = w_0 r_\theta t_{-\theta^\vee} w_0 = r_\theta t_{\theta^\vee} = r_0 t_{2\theta^\vee}$. Therefore for every $x \in W_{\mathrm{af}}$, there is a $\mu \in Q^\vee$ such that $w_0 x w_0 = x^* t_\mu$. Using (8) and $w_0^P = (w_0^P)^{-1}$ we have

$$w_0 \tau_j(y) w_0 = w_0 \tau_j y \tau_j^{-1} w_0 = w_0^P v_j^{-1} \tau_j y \tau_j^{-1} v_j w_0^P = w_0^P t_{-\omega_j^\vee} y t_{\omega_j^\vee} w_0^P = w_0^P y w_0^P t_\mu$$

for some $\mu \in Q^\vee$. Thus $\vartheta(y) = w_0^P y w_0^P t_\lambda$ for some $\lambda \in Q^\vee$. But clearly we have $\pi_P(w_0^P y w_0^P) = \pi_P(w_0^P y)$, giving us the last statement of the proposition.

The map $y \mapsto w_0^P y$ induces an involution on W^P. Since σ_P^y is a $S[q, q^{-1}]$-basis of $QH^T(G/P)[q^{-1}]$ we conclude by Theorem 10.16 that $\xi_{\vartheta(y)}$ is a $S[\xi_{\pi_P(t_\lambda)}^{\pm 1} \mid \lambda \in \tilde{Q}]$-basis of $(H_T(\mathrm{Gr}_G)/J_P)[\xi_{\pi_P(t_\lambda)}^{-1} \mid \lambda \in \tilde{Q}]$. □

Remark 11.1. Since it is defined using automorphisms of the affine Dynkin diagram, the map ϑ induces an isomorphism of the Bruhat order on W^P with that on its image.

Example 11.1. Let $G = SL(7)$, $j = 4$, and $y = r_4 r_5 r_2 r_3 r_4 \in W^P$, which in one-line notation (that is, the list $y(1), y(2), \ldots, y(7)$, viewing y as a permutation of $\{1, 2, \ldots, 7\}$) is $y = (1356 \mid 247)$ and therefore corresponds to the partition $(6, 5, 3, 1) - (4, 3, 2, 1) = (2, 2, 1, 0)$ inside the 4×3 rectangle. The above reduced decomposition of y is obtained by the columnwise reading of simple reflections in the following picture of the French diagram of the $(2, 2, 1, 0)$, where the cell (x_1, x_2) contains the value $j + x_1 - x_2$ where the lower left cell is indexed $(1, 1)$ and the cells are indexed by integer lattice points in the first quadrant of the Cartesian plane. In general if μ is the partition denote the corresponding element of W^P by w_μ.

$$\begin{array}{|c|c} \hline 2 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{4} \\ \hline 4 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad w_{(2,2,1,0)} = r_4 r_5 r_2 r_3 r_4.$$

We have $\tau_j(y) = r_0 r_1 r_5 r_6 r_0$ and $\vartheta(y) = r_0 r_6 r_2 r_1 r_0 = w t_\lambda$ where $\lambda = -\omega_2^\vee - \omega_5^\vee$ and $w = r_\theta r_6 r_2 r_1 r_\theta$, which in one-line notation is $w = (6724 \mid 513)$. Then $\pi_P(t_\lambda) = r_2 r_3 r_1 r_2 r_6 r_5 t_\lambda$ and $\pi_P(w) = (2467|135)$ which corresponds to the partition $(7, 6, 4, 2) - (4, 3, 2, 1) = (3, 3, 2, 1)$.

11.3. **Strange duality.** In [6], Chaput, Manivel and Perrin study a *strange duality* involution on $QH^*(G/P)[q, q^{-1}]$. The final statement of Proposition 11.3 suggests a relationship between strange duality and Theorem 10.16.

Theorem 11.4 ([6, Theorem 4.1])**.** *Let $P \subset G$ be a cominuscule parabolic subgroup with $I_P = I \setminus \{j\}$ and for $w \in W^P$ let $\delta(w)$ be the number of times r_j appears in some (and thus any) reduced decomposition of w. Then there exists a function $\zeta : W \to \mathbb{R}$ such that*

$$q \mapsto q^{-1} \qquad \sigma_P^w \mapsto \zeta(w) q^{-\delta(w)} \sigma_P^{\pi_P(w_0^P w)}$$

defines an involutive ring automorphism of $QH^*(G/P)[q^{-1}] \otimes_{\mathbb{Z}} \mathbb{R}$.

In general the values $\zeta(w)$ can be irrational algebraic numbers, but for $G = SL(n)$, $\zeta(w) = 1$ for all $w \in W$.

One may check that Example 11.1 agrees with the explicit description in [6] of strange duality on the Grassmannian in terms of partitions and their Durfee square.

11.4. **The homomorphism of Lapointe and Morse.** Suppose now that G/P is the Grassmannian $\mathrm{Gr}(j, \mathbb{C}^n) = SL_n/P$. Lapointe and Morse defined a map which, after various identifications, can be interpreted as a surjective ring homomorphism $H_*(\mathrm{Gr}_{SL_n}) \to QH^*(\mathrm{Gr}(j, \mathbb{C}^n))$. We shall explain their map in terms of strange duality and the parabolic Peterson Theorem (Theorem 10.16).

For this section let $k = n-1$. In [17], motivated by Macdonald theory, Lapointe, Lascoux, and Morse defined a family of symmetric functions $s_\lambda^{(k)}$ called k-Schur functions. They form a basis of the ring $\mathbb{Z}[h_1, \ldots, h_k]$, where h_i is the homogeneous symmetric function. The k-Schur basis is indexed by k-bounded partitions, that is, partitions λ such that $\lambda_1 \leq k$.

The homomorphism of Lapointe and Morse may be described as follows.

Theorem 11.5. [19] *There is a surjective ring homomorphism $\mathbb{Z}[h_1, \ldots, h_{n-1}] \to QH^*(\mathrm{Gr}(j, \mathbb{C}^n))$ such that for any $(n-1)$-bounded partition λ, the $(n-1)$-Schur function $s_\lambda^{(n-1)}$ maps to 0 or a power of q times a single quantum Schubert class. Moreover,*

1. *If λ fits inside the $(n-j) \times j$ rectangle then $s_\lambda^{(n-1)} \mapsto \sigma_P^{w_{\lambda^t}}$ where λ^t is the transpose of the partition λ.*
2. *If $\lambda_1 > j$ then $s_\lambda^{(n-1)} \mapsto 0$.*

The above rules specify the map except when λ consists of some number of parts of size j followed by a partition contained in the $(n-j) \times j$ rectangle; in that case one must use a straightening process to determine the image Schubert class explicitly; see [19].

Bott [5] gave an explicit realization of $H_*(\mathrm{Gr}_{SL_n})$ by the ring $\mathbb{Z}[h_1, \ldots, h_{n-1}]$. Lam [14] proved that the $(n-1)$-Schur functions are the Schubert basis of $H_*(\mathrm{Gr}_{SL_n})$. To make the identification explicit, we recall a bijection [18, Proposition 47] denoted here by $\lambda \mapsto w_\lambda^{\mathrm{af}}$, from $(n-1)$-bounded partitions W_{af}^-, where W_{af} is the affine Weyl group for $G = SL_n$. See [15] for alternative descriptions of this bijection.

Given the $(n-1)$-bounded partition λ, we place the value $x_1 - x_2 \mod n$ into the cell (x_1, x_2) in the diagram of λ in a manner similar to the definition of w_λ in Example 11.1.

These entries are then used as indices for simple reflections in a reduced decomposition of an element $w_\lambda^{\mathrm{af}} \in W_{\mathrm{af}}^-$, reading the rows in order from the top row to the bottom row, reading within each row from right to left.

Example 11.2. Let $n = 7$ and $\lambda = (3, 2)$. Then the filled diagram of λ is given by

6	0	
0	1	2

and $w_\lambda^{\mathrm{af}} = r_0 r_6 r_2 r_1 r_0$.

Theorem 11.6. [14] *Under Bott's isomorphism $H_*(\mathrm{Gr}_{SL_n}) \cong \mathbb{Z}[h_1, \ldots, h_{n-1}]$, the Schubert class $\xi_{w_\lambda^{\mathrm{af}}}$ maps to the $(n-1)$-Schur function $s_\lambda^{(n-1)}$ for every $(n-1)$-bounded partition λ.*

Combining Theorem 11.5 specialized at $q = 1$ and Theorem 11.6 one obtains the Lapointe-Morse ring homomorphism $\Psi_{LM} : H_*(\mathrm{Gr}_{SL_n}) \to QH^*(\mathrm{Gr}(j, \mathbb{C}^n))|_{q=1}$.

On the other hand, combining strange duality and the parabolic Peterson Theorem we have the following result.

Proposition 11.7. *Let $G = SL_n$ and $P \subset G$ be a maximal parabolic subgroup with $I_P = I \setminus \{j\}$. Then there is a surjective ring homomorphism $\Psi : H_*(\mathrm{Gr}_G) \to QH^*(G/P)|_{q=1}$ defined by*

$$\xi_x \mapsto \begin{cases} \sigma_P^y & \text{if } x = \vartheta(y)\pi_P(t_\lambda) \text{ for some } y \in W^P \text{ and } \lambda \in Q^\vee, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover $\Psi = \Psi_{LM}$.

Proof. Ψ is the composition of the nonequivariant specialization of the map of Theorem 10.16 and the map of Theorem 11.4 specialized at $q = 1$. Together with Proposition 11.3 it follows that Ψ is a surjective ring homomorphism.

To prove $\Psi = \Psi_{LM}$ it suffices to check agreement on algebra generators. For $0 \le m \le n-1$ let $h_{[m]} = r_{m-1} \cdots r_2 r_1 r_0 \in W_{\mathrm{af}}^-$; $\xi_{h_{[m]}}$ is the Bott generator corresponding to the symmetric function h_m and to the $(n-1)$-bounded partition having a single row of size m.

Let $y \in W^P$. Let λ be the partition contained in the $(n-j) \times j$ rectangle such that $y = w_{\lambda^t}$. It is easy to check from the definitions that $\vartheta(y) = w_\lambda^{\mathrm{af}}$. Consequently Ψ and Ψ_{LM} agree on $\xi_{\vartheta(y)}$ for $y \in W^P$. Since W^P contains the elements $c_{[m]} = r_{j-m+1} \cdots r_{j-1} r_j$ for $0 \le m \le j$ and $\vartheta(c_{[m]}) = h_{[m]}$ for $0 \le m \le j$, $\Psi = \Psi_{LM}$ on the generators $\xi_{h_{[m]}}$ for $0 \le m \le j$. Finally, both Ψ and Ψ_{LM} send $\xi_{h_{[m]}}$ to zero for $j+1 \le m \le n-1$. □

Example 11.3. Let $n = 7$, $j = 4$ and choose $\lambda = (3, 2, 0)$ in the 3×4 rectangle. Then $\lambda^t = (2, 2, 1, 0)$ fits in the 4×3 rectangle and $w_{\lambda^t} \in W^P$ is given by the element y of Example 11.1. The element $\vartheta(y)$ is given by w_λ^{af}, which appears in the two previous examples.

Remark 11.2. The "Pieri formula" for $H_*(\mathrm{Gr}_{SL_n})$ was given in [15], and agrees with the k-Pieri rule of Lapointe and Morse [18]. The image of this Pieri rule under Ψ is exactly the quantum Pieri rule of $QH^*(\mathrm{Gr}(j, \mathbb{C}^n))$; see [1].

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